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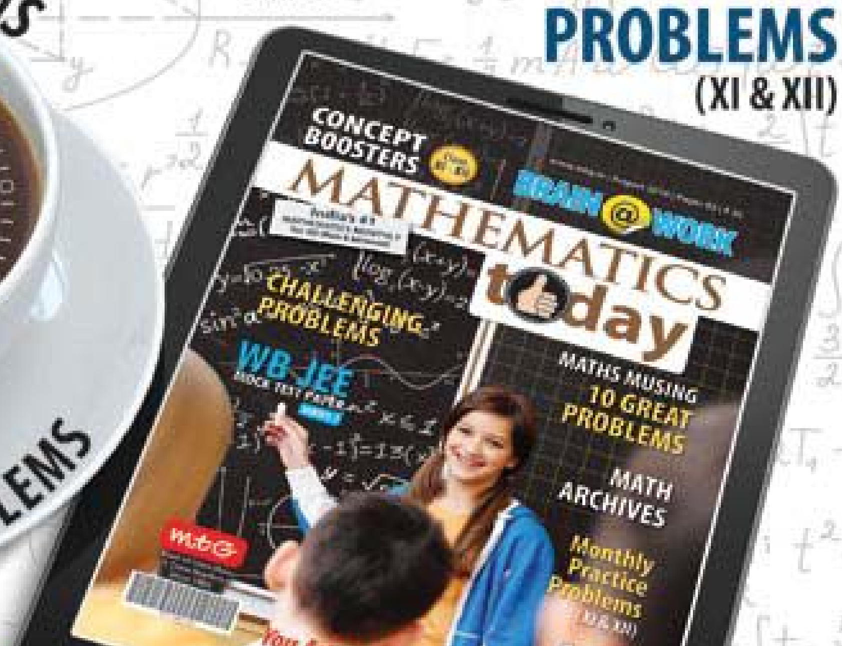

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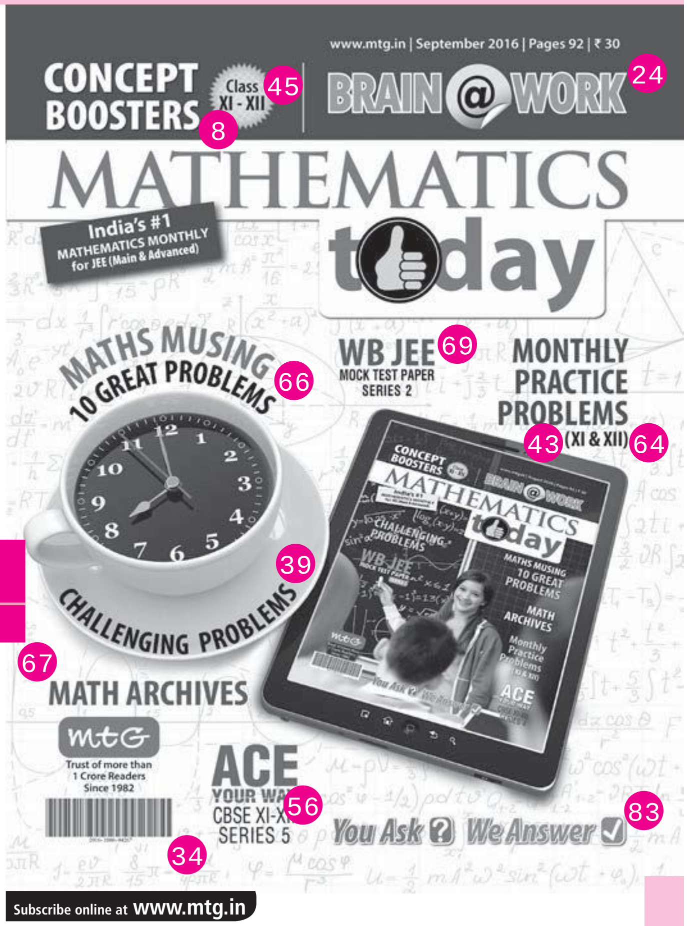

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Editor : Anil Ahlawat

CONCEPT BOOSTERS

Class XI

COMPLEX NUMBERS

This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

THE COMPLEX NUMBER SYSTEM

(a) Solutions of equation $x^2 + 1 = 0$ are not real so they are imaginary. i was regarded as a fictitious or imaginary number which could be manipulated algebrically like an ordinary real number, except that its square was –1. The letter i was used to denote $\sqrt{-1}$, possibly because i is the first letter of the Latin word 'imaginarius'.

(b) To permit solutions of such polynomial equations, the set of complex numbers is introduced. We can consider a complex number of the form $a + ib$, where a and b are real numbers. It is denoted by z *i.e.* $z = a + ib$. 'a' is called as real part of z which is denoted by Re(z) and 'b' is called as imaginary part of z which is denoted by Im (z).

Every complex number can be regarded as

Purely real	Purely Imaginary	Imaginary
if $b = 0$	if $a = 0$	if $b \neq 0$

Remarks :

(a) The set R of real numbers is a proper subset of the Complex Numbers. Hence the complete number system is $N \subset W \subset I \subset Q \subset R \subset C$.

(b) Zero is purely real as well as purely imaginary.

(c) $i = \sqrt{-1}$ is the imaginary unit and called 'iota'. Also $i^2 = -1$; $i^3 = -i$; $i^4 = 1$ etc.

(d) $\sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$ only if atleast one of a or b is non-negative.

(e) If $z = a + ib$, then $a - ib$ is called complex conjugate of z and written as $\bar{z} = a - ib$.

(f) Real numbers satisfy order relations whereas imaginary numbers do not satisfy order relation *i.e.* $i > 0$, $3 + i < 2$ are meaningless.

ALGEBRAIC OPERATIONS

- **Fundamental operations on complex numbers**

  In performing operations with complex numbers we can proceed as in the algebra of real numbers, replacing i^2 by –1 when it occurs.

  (i) Addition :

  $(a + bi) + (c + di) = a + bi + c + di$
  $= (a + c) + (b + d)i$

  (ii) Subtraction :

  $(a + bi) - (c + di) = a + bi - c - di$
  $= (a - c) + (b - d)i$

  (iii) Multiplication :

  $(a + bi)(c + di) = ac + adi + bci + bdi^2$
  $= (ac - bd) + (ad + bc)i$

  (iv) Division :

  $$\frac{a+bi}{c+di} = \frac{a+bi}{c+di}\cdot\frac{c-di}{c-di} = \frac{ac-adi+bci-bdi^2}{c^2-d^2i^2}$$
  $$= \frac{ac+bd+(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

- Inequalities in imaginary numbers are not defined. There is no validity if we say that imaginary number is positive or negative.

  e.g., $z > 0$, $4 + 2i < 2 + 4i$ are meaningless.

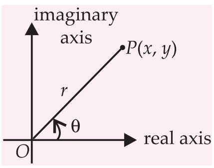
  In real numbers if $a^2 + b^2 = 0$ then $a = 0 = b$ however in complex numbers,

  $z_1^2 + z_2^2 = 0$ does not imply $z_1 = 0 = z_2$.

- **Equality In Complex Numbers :** Two complex numbers $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$ are equal if and only if their real and imaginary parts are equal
  i.e. $z_1 = z_2 \iff \text{Re}(z_1) = \text{Re}(z_2)$
  and $\text{Im}(z_1) = \text{Im}(z_2)$.

REPRESENTATION OF A COMPLEX NUMBER

- **Cartesian Form (Geometric Representation)**
  To each complex number there corresponds one and only one point in plane, and conversely to each point in the plane there corresponds one and only one complex number. Because of this we often refer to the complex number z as the point z.
  Every complex number $z = x + iy$ can be represented by a point on the Cartesian plane known as complex plane (Argand diagram) by the ordered pair (x, y).



Length OP is called modulus of the complex number which is denoted by $|z|$ & θ is called the argument or amplitude.

$|z| = \sqrt{x^2 + y^2}$ and $\tan\theta = \left(\dfrac{y}{x}\right)$ (angle made by OP with positive x-axis)

Note :

(i) Argument of a complex number is a many valued function. If θ is the argument of a complex number then $2n\pi + \theta$; $n \in I$ will also be the argument of that complex number. Any two consecutive arguments of a complex number differ by $2n\pi$.

(ii) The unique value of θ such that $-\pi < \theta \le \pi$ is called the principal value of the argument. Unless otherwise stated, amp z implies principal value of the argument.

(iii) By specifying the modulus & argument a complex number is defined completely. For the complex number $0 + 0i$ the argument is not defined and this is the only complex number which is only given by its modulus.

- **Trigonometric/Polar Representation**

$$z = r(\cos\theta + i\sin\theta), \text{ where } |z| = r;\ \arg z = \theta;$$
$$\bar{z} = r(\cos\theta - i\sin\theta)$$

Note : $\cos\theta + i\sin\theta$ is also written as $CiS\ \theta$

- **Euler's Formula**

$$z = re^{i\theta},\ |z| = r,\ \arg z = \theta$$
$$\bar{z} = re^{-i\theta}$$

Note : If θ is real then

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2};\ \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

- **Vectorial Representation**
  Every complex number can be considered as the position vector of a point. If the point P represents the complex number z, then

$$\overrightarrow{OP} = z \text{ and } |\overrightarrow{OP}| = |z|$$

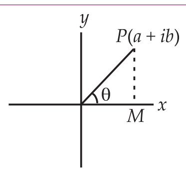
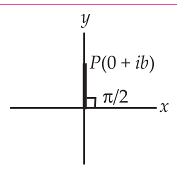
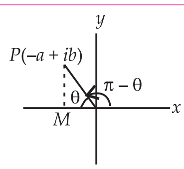
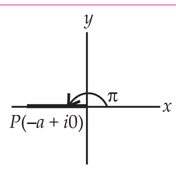
ARGUMENT OF A COMPLEX NUMBER

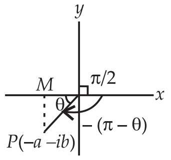
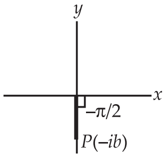
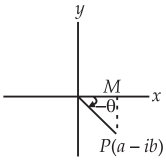
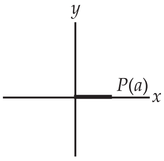
(a) Argument of a non-zero complex number $P(z)$ is denoted and defined by arg (z) = angle which OP makes with the positive direction of real axis.

(b) If $OP = |z| = r$ and arg $(z) = \theta$, then obviously $z = r(\cos\theta + i\sin\theta)$, called the polar form of z. 'Argument of z' would mean principal argument of z(*i.e.* argument lying in $(-\pi, \pi]$) unless the context requires otherwise.

(c) Argument of a complex number $z = a + ib$ $= r(\cos\theta + i\sin\theta)$ is the value of θ satisfying $r\cos\theta = a$ and $r\sin\theta = b$.

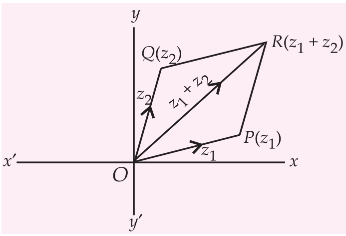
$$\text{Let } \theta = \tan^{-1}\left|\frac{b}{a}\right|$$

1.	$a > 0,\ b > 0$		P.V. $\arg z = \theta$
2.	$a = 0,\ b > 0$		P.V. $\arg z = \pi/2$
3.	$a < 0,\ b > 0$		P.V. $\arg z = \pi - \theta$
4.	$a < 0,\ b = 0$		P.V. $\arg z = \pi$

5.	$a < 0, b < 0$		P.V. $\arg z = -(\pi - \theta)$
6.	$a = 0, b < 0$		P.V. $\arg z = -\pi/2$
7.	$a > 0, b < 0$		P.V. $\arg z = -\theta$
8.	$a > 0, b = 0$		P.V. $\arg z = 0$
	P.V. stands for principal value		

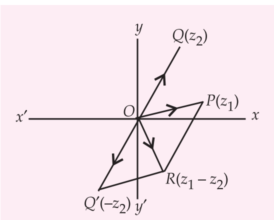
GEOMETRICAL REPRESENTATION OF FUNDAMENTAL OPERATIONS

(a) Geometrical representation of addition of complex numbers



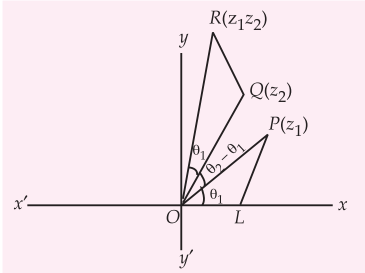
If two points P and Q represent complex numbers z_1 and z_2 respectively in the Argand plane, then the sum $z_1 + z_2$ is represented by the extremity R of the diagonal OR of parallelogram $OPRQ$ having OP and OQ as two adjacent sides.

(b) Geometrical representation of subtraction of complex numbers



(c) Geometrical representation of multiplication of complex numbers

Let P, Q be represented by $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$ respectively. To find point R representing complex number z_1z_2, we take a point L on real axis such that $OL = 1$ and draw triangle OQR similar to triangle OLP. Therefore



$$\frac{OR}{OQ} = \frac{OP}{OL} \Rightarrow OR = OP \cdot OQ$$

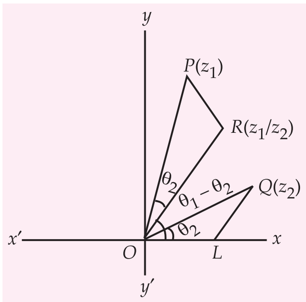
i.e. $OR = r_1r_2$ and $\angle QOR = \theta_1$

$\angle LOR = \angle LOP + \angle POQ + \angle QOR$
$= \theta_1 + \theta_2 - \theta_1 + \theta_1 = \theta_1 + \theta_2$

Hence, R is represented by $z_1z_2 = r_1r_2 e^{i(\theta_1+\theta_2)}$

(d) Geometrical representation of the division of complex numbers

Let points P, Q be represented by $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$ respectively. To find point R representing complex number $\frac{z_1}{z_2}$, we take a point L on real axis such that $OL = 1$ and draw a triangle OPR similar to OQL.



Therefore $\frac{OP}{OQ} = \frac{OR}{OL} \Rightarrow OR = \frac{r_1}{r_2}$

and $\angle LOR = \angle LOP - \angle ROP = \theta_1 - \theta_2$

Hence, R is represented by $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

- **Modulus and argument of multiplication of two complex numbers**
  For any two complex numbers z_1, z_2, we have $|z_1z_2| = |z_1||z_2|$ and
  $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$.

Notes :

(i) P.V. $\arg(z_1z_2) \neq$ P.V. $\arg(z_1)$ + P.V. $\arg(z_2)$

(ii) $|z_1\, z_2 \,....\, z_n| = |z_1||z_2| \,.....\, |z_n|$

(iii) $\arg(z_1z_2 \,....\, z_n) = \arg z_1 + \arg z_2 + \,.....\, + \arg z_n$

- **Modulus and argument of division of two complex numbers.**

  If z_1 and z_2 ($\neq 0$) are two complex numbers, we have $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ and

  $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

  Note :

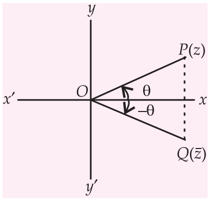
  P.V. $\arg\left(\dfrac{z_1}{z_2}\right) \neq$ P.V. $\arg(z_1) -$ P.V. $\arg(z_2)$

CONJUGATE OF A COMPLEX NUMBER

- Conjugate of a complex number $z = a + ib$ is denoted and defined by $\bar{z} = a - ib$.
  In a complex number if we replace i by $-i$, we get conjugate of the complex number. $\bar{z}$ is the mirror image of z about real axis on Argand's Plane.
- **Geometrical representation of conjugate of complex number :**

  $|z| = |\bar{z}|$

  $\arg(\bar{z}) = -\arg(z)$



- **Properties**

  (i) If $z = x + iy$, then $x = \dfrac{z+\bar{z}}{2}$, $y = \dfrac{z-\bar{z}}{2i}$

  (ii) $z = \bar{z} \iff z$ is purely real.

  (iii) $z + \bar{z} = 0 \iff z$ is purely imaginary.

  (iv) $|z|^2 = z\bar{z}$

  (v) $\bar{\bar{z}} = z$

  (vi) $\overline{(z_1 \pm z_2)} = \bar{z}_1 \pm \bar{z}_2$

  (vii) $\overline{(z_1\, z_2)} = \bar{z}_1\bar{z}_2$

  (viii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{(\bar{z}_1)}{(\bar{z}_2)}, (z_2 \neq 0)$

- Imaginary roots of polynomial equations with real coefficients occur in conjugate pairs.
- $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm (z_1\bar{z}_2 + \bar{z}_1z_2)$

  $= |z_1|^2 + |z_2|^2 \pm 2\,\mathrm{Re}(z_1\bar{z}_2)$

  $= |z_1|^2 + |z_2|^2 \pm 2|z_1||z_2| \cos(\theta_1 - \theta_2)$

  Note : If $w = f(z)$, then $\bar{w} = f(\bar{z})$

DISTANCE BETWEEN COMPLEX POINTS

If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, then distance between points z_1, z_2 in argand plane is

$|z_1 - z_2| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

INEQUALITIES IN COMPLEX NUMBERS

If z_1 and z_2 are two complex numbers such that $|z_1| \leq 1$, $|z_2| \leq 1$, then

(i) $|z_1 - z_2|^2 \leq (|z_1| - |z_2|)^2 + (\mathrm{Arg}(z_1) - \mathrm{Arg}(z_2))^2$

(ii) $|z_1 + z_2|^2 \geq (|z_1| + |z_2|)^2 - (\mathrm{Arg}(z_1) - \mathrm{Arg}(z_2))^2$

(iii) $|z_1 \pm z_2| \leq |z_1| + |z_2|$

In general $|z_1 \pm z_2 \pm z_3 \pm \,.....\, \pm z_n| \leq |z_1| + |z_2| + |z_3| + \,.....\, + |z_n|$

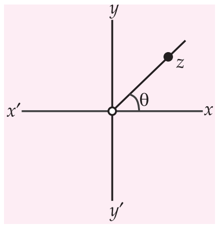
(iv) $|z_1 \pm z_2| \geq ||z_1| - |z_2||$

$\Rightarrow ||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

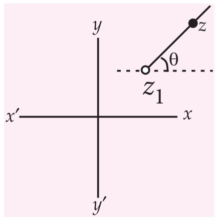
ROTATION

(a) Important results

(i) $\arg z = \theta$ represents points (non-zero) on ray eminating from origin making an angle θ with positive direction of real axis.

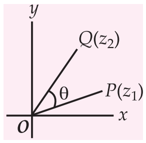


(ii) $\arg(z - z_1) = \theta$ represents points ($\neq z_1$) on ray eminating from z_1 making an angle θ with positive direction of real axis.

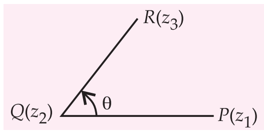


(b) Rotation theorem

(i) If $P(z_1)$ and $Q(z_2)$ are two complex numbers such that $|z_1| = |z_2|$, then $z_2 = |z_1|\, e^{i\theta}$ where $\theta = \angle POQ$

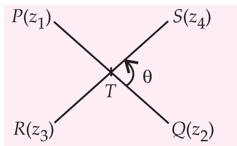


(ii) If $P(z_1)$, $Q(z_2)$ and $R(z_3)$ are three complex numbers and $\angle PQR = \theta$, then



$\left(\dfrac{z_3 - z_2}{z_1 - z_2}\right) = \left|\dfrac{z_3 - z_2}{z_1 - z_2}\right| e^{i\theta}$

(iii) If $P(z_1)$, $Q(z_2)$, $R(z_3)$ and $S(z_4)$ are four complex numbers and $\angle STQ = \theta$, then



$\dfrac{z_3 - z_4}{z_1 - z_2} = \left|\dfrac{z_3 - z_4}{z_1 - z_2}\right| e^{i\theta}$

DE MOIVRE'S THEOREM

(a) Case I :

Statement : If n is any integer, then

(i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(ii) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$
$(\cos\theta_3 + i\sin\theta_3)\ldots..(\cos\theta_n + i\sin\theta_n)$
$= \cos(\theta_1 + \theta_2 + \theta_3 + \ldots\ldots\theta_n)$
$+ i\sin(\theta_1 + \theta_2 + \theta_3 + \ldots. + \theta_n)$

(b) Case II :

Statement : If $p, q \in Z$ and $q \neq 0$ then

$$(\cos\theta + i\sin\theta)^{p/q} = \cos\left(\frac{2k\pi + p\theta}{q}\right) + i\sin\left(\frac{2k\pi + p\theta}{q}\right)$$

where $k = 0, 1, 2, 3, \ldots., q - 1$

Note : Continued product of the roots of a complex quantity should be determined using theory of equations.

CUBE ROOTS OF UNITY

- The cube roots of unity are
$$1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$$
- If ω is one of the imaginary cube roots of unity then $1 + \omega + \omega^2 = 0$. In general $1 + \omega^r + \omega^{2r} = 0$; where $r \in I$ but is not the multiple of 3.
- In polar form the cube roots of unity are :
$$\cos 0 + i\sin 0;\ \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3},$$
$$\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$
- The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.
- The following factorisations should be remembered : ($a, b, c \in R$ & ω is the cube root of unity)

(i) $a^3 \pm b^3 = (a \pm b)(a \pm \omega b)(a \pm \omega^2 b)$;

(ii) $x^2 \pm x + 1 = (x \mp \omega)(x \mp \omega^2)$;

(iii) $a^2 \pm ab + b^2 = (a \mp b\omega)(a \mp b\omega^2)$

(iv) $a^3 + b^3 + c^3 - 3abc = (a + b + c)$
$(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

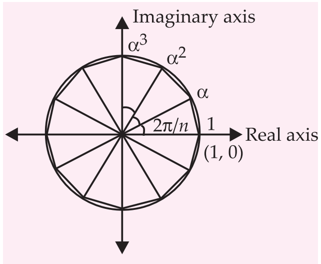
(v) $a^2 + b^2 + c^2 - ab - bc - ca$
$= (a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

(vi) $a^2 + b^2 = (a + ib)(a - ib)$

n^{th} ROOTS OF UNITY

If $1, \alpha_1, \alpha_2, \alpha_3\ldots.. \alpha_{n-1}$ are the n, n^{th} roots of unity, then :

- They are in G.P. with common ratio $e^{i(2\pi/n)}$



- $1^p + \alpha_1^p + \alpha_2^p + \ldots.. + \alpha_{n-1}^p$
$$= \begin{cases} 0, & \text{if } p \text{ is not an integral multiple of } n \\ n, & \text{if } p \text{ is an integral multiple of } n \end{cases}$$
- $(1 - \alpha_1)(1 - \alpha_2)\ldots..(1 - \alpha_{n-1}) = n$ and $(1 + \alpha_1)(1 + \alpha_2)\ldots..(1 + \alpha_{n-1})$
$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$$
- $1 \cdot \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot\ldots.. \alpha_{n-1} = 1$ or -1 according as n is odd or even.

TWO IMPORTANT SERIES

- $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots.. + \cos n\theta$
$$= \frac{\sin(n\theta/2)}{\sin(\theta/2)}\cos\left(\frac{n+1}{2}\right)\theta$$
- $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots.. + \sin n\theta$
$$= \frac{\sin(n\theta/2)}{\sin(\theta/2)}\sin\left(\frac{n+1}{2}\right)\theta$$

Note : If $\theta = (2\pi/n)$, then the sum of the above series vanishes.

LOGARITHM OF A COMPLEX QUANTITY

(a) $$\log_e(\alpha + i\beta) = \frac{1}{2}\log_e(\alpha^2 + \beta^2) + i\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right)$$
where $n \in I$.

(b) i^i represents a set of positive real numbers given by $e^{-\left(2n\pi + \frac{\pi}{2}\right)}$, $n \in I$

GEOMETRICAL PROPERTIES

(a) Section formulae : If a point C divides the line segment joining $P(z_1)$ and $Q(z_2)$ internally in the ratio $m : n$, then affix z of C is given by
$$z = \frac{mz_2 + nz_1}{m + n}$$
If C divides PQ in the ratio $m : n$ externally, then affix z of C is given by $z = \dfrac{mz_2 - nz_1}{m - n}$

If C is the mid-point of PQ, then affix z of C is given by $z = \frac{z_1 + z_2}{2}$

Remark : If a, b, c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$; where $a + b + c = 0$ and a, b, c are not all simultaneously zero, then the complex numbers z_1, z_2 & z_3 are collinear.

(b) If the vertices A, B, C of a Δ are represented by complex numbers z_1, z_2, z_3 respectively and a, b, c are the length of sides, then

(i) Centroid of $\Delta ABC = \frac{z_1 + z_2 + z_3}{3}$

(ii) Orthocentre of ΔABC

$$= \frac{(a \sec A) z_1 + (b \sec B) z_2 + (c \sec C) z_3}{a \sec A + b \sec B + c \sec C}$$

or $\frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\tan A + \tan B + \tan C}$

(iii) Incentre of ΔABC
$= (az_1 + bz_2 + cz_3) \div (a + b + c)$.

(iv) Circumcentre of $\Delta ABC =$
$(z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C)$
$\div (\sin 2A + \sin 2B + \sin 2C)$.

(c) amp $(z) = \theta$ is a ray emanating from the origin inclined at an angle θ to the x-axis.

(d) $|z - a| = |z - b|$ is the perpendicular bisector of the line joining a to b.

(e) The equation of a line joining z_1 & z_2 is given by, $z = z_1 + t(z_1 - z_2)$, where t is a real parameter.

(f) $z = z_1 (1 + it)$, where t is a real parameter is a line through the point z_1 & perpendicular to the line joining z_1 to the origin.

(g) The equation of a line passing through z_1 & z_2 can be expressed in the determinant form as

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0.$$

This is also the condition for three complex numbers z, z_1, z_2 to be collinear.

(h) The equation of circle having centre z_0 & radius ρ is :

$|z - z_0| = \rho$ or $z\bar{z} - z_0\bar{z} - \bar{z}_0 z + \bar{z}_0 z_0 - \rho^2 = 0$ which is of the form $z\bar{z} + \bar{\alpha} z + \alpha\bar{z} + k = 0$, k is real.

Centre is $-\alpha$ & radius $= \sqrt{\alpha\bar{\alpha} - k}$

Circle will be real if $\alpha\bar{\alpha} - k \geq 0$.

(i) The equation of the circle described on the line segment joining z_1 & z_2 as diameter is

$$\arg \frac{z - z_2}{z - z_1} = \pm \frac{\pi}{2}$$

or $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$

(j) Condition for four given points z_1, z_2, z_3 & z_4 to be concyclic is the number $\frac{z_3 - z_1}{z_3 - z_2} \cdot \frac{z_4 - z_2}{z_4 - z_1}$ should be real. Hence the equation of a circle through 3 non collinear points z_1, z_2 & z_3 can be taken as $\frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)}$ is real

$$\Rightarrow \frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)} = \frac{(\bar{z} - \bar{z}_2)(\bar{z}_3 - \bar{z}_1)}{(\bar{z} - \bar{z}_1)(\bar{z}_3 - \bar{z}_2)}$$

(k) $\text{Arg}\left(\frac{z - z_1}{z - z_2}\right) = \theta$ represent

(i) a line segment, if $\theta = \pi$
(ii) a pair of ray, if $\theta = 0$
(iii) a part of circle, if $0 < \theta < \pi$.

(l) Area of triangle formed by the points z_1, z_2 & z_3 is $\left| \frac{1}{4i} \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} \right|$

(m) Perpendicular distance of a point z_0 from the line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ is $\frac{|\bar{\alpha}z_0 + \alpha\bar{z}_0 + r|}{2|\alpha|}$

(n) General equation of a straight line is given by $\alpha\bar{z} + \bar{\alpha}z + r = 0$, where α is a complex number and r is real number

(i) Real slope of a line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ ($\alpha \in z$, a complex number $r \in R$) is given by $-\frac{\alpha}{\bar{\alpha}} = \frac{-\text{coeff. of } \bar{z}}{\text{coeff. of } z}$

(ii) Slope of a line segment joining the points z_1 & z_2 is given by $\omega = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$

(iii) Complex slope of the line $\alpha\bar{z} + \bar{\alpha}z + r = 0$ ($r \in R$, $\alpha \in Z$ a complex number) is given by $-\frac{\bar{\alpha}}{\alpha} = \frac{-\text{coeff.of } \bar{z}}{\text{coeff.of } z}$

(iv) Complex slope of a line making angle θ with real axis is $\omega = e^{2i\theta}$

(o) Dot and cross product

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers [vectors]. The dot product [also called the scalar product] of z_1 and z_2 is defined by

$z_1 \cdot z_2 = |z_1||z_2|\cos\theta = x_1x_2 + y_1y_2$

$$= \text{Re}\{\bar{z}_1 z_2\} = \frac{1}{2}\{\bar{z}_1 z_2 + z_1 \bar{z}_2\}$$

where θ is the angle between z_1 and z_2 which lies between 0 and π.

If vectors z_1, z_2 are perpendicular then

$$z_1 \cdot z_2 = 0 \Rightarrow \frac{z_1}{\bar{z}_1} + \frac{z_2}{\bar{z}_2} = 0$$

i.e. Sum of complex slopes = 0

The cross product of z_1 and z_2 is defined by

$z_1 \times z_2 = |z_1||z_2| \sin\theta = x_1y_2 - y_1x_2$

$$= \text{Im}\{\bar{z}_1 z_2\} = \frac{1}{2i}\{\bar{z}_1 z_2 - z_1 \bar{z}_2\}$$

If vectors z_1, z_2 are parallel then $z_1 \times z_2 = 0$

$$\Rightarrow \frac{z_1}{\bar{z}_1} = \frac{z_2}{\bar{z}_2}$$

i.e. Complex slopes are equal.

PTOLEMY'S THEOREM

It states that the product of the lengths of the diagonals of a convex quadrilateral inscribed in a circle is equal to the sum of the products of lengths of the two pairs of its opposite sides.

i.e. $|z_1 - z_3|\,|z_2 - z_4| = |z_1 - z_2|\,|z_3 - z_4| + |z_1 - z_4|\,|z_2 - z_3|$.

PROBLEMS

Single Correct Answer Type

1. The equation of the circle whose centre is at $a + i$ (where a is a real number) and intersecting two circles $|z| = 1$ and $|z - 1| = 4$ orthogonally is

(a) $|z - 7 + i| = 7$ (b) $|z + 7 - i| = 7$

(c) $|z - 2 + i| = 7$ (d) $|z + 2 - i| = 7$

2. If 1, α_1, α_2, ..., α_{n-1} are n^{th} roots of unity then $\frac{1}{1-\alpha_1} + \frac{1}{1-\alpha_2} + ... + \frac{1}{1-\alpha_{n-1}} =$

(a) $\frac{n-1}{2}$ (b) $\frac{n}{2}$ (c) $\frac{2^n - 1}{2}$ (d) $\frac{n-1}{2n}$

3. Modulus of non-zero complex number z satisfying $\bar{z} + z = 0$ and $|z|^2 - 4zi = z^2$ is

(a) 1 (b) 2 (c) 3 (d) 4

4. If z and ω are two non-zero complex numbers such that $|z\omega| = 1$ and $\text{Arg}\, z - \text{Arg}\, \omega = \frac{\pi}{2}$, then $\bar{z}\omega =$

(a) 1 (b) -1 (c) i (d) $-i$

5. If ω is a complex number such that $|\omega| = r \neq 1$ then $z = \omega + \frac{1}{\omega}$ describes a conic. The distance between the foci is

(a) 2 (b) $2(\sqrt{2} - 1)$

(c) 3 (d) 4

6. The number of solutions of the system of equations given by $|z| = 3$ and $|z + 1 - i| = \sqrt{2}$ is equal to

(a) 4 (b) 2

(c) 1 (d) no solution

7. If z_1, z_2 and z_3 be the vertices of ΔABC, taken in anti-clock wise direction and z_0 be the circumcentre, then $\left(\frac{z_0 - z_1}{z_0 - z_2}\right)\frac{\sin 2A}{\sin 2B} + \left(\frac{z_0 - z_3}{z_0 - z_2}\right)\frac{\sin 2C}{\sin 2B}$ is equal to

(a) 0 (b) 1 (c) -1 (d) 2

8. If $z = x + iy$ such that $|z - 4| < |z - 2|$, then

(a) $x > 0, y > 0$

(b) $x < 0, y > 0$

(c) $x > 2, y > 3$

(d) $x > 3$ and y is any real number

9. If z_1, z_2 are complex numbers such that $z_1^2 + z_2^2$ is real. If $z_1(z_1^2 - 3z_2^2) = 2$ and $z_2(3z_1^2 - z_2^2) = 11$, then the value of $z_1^2 + z_2^2 =$

(a) 25 (b) 5 (c) $\sqrt{5}$ (d) 1

10. If z_1 and z_2 are two complex numbers satisfying $\frac{z_1}{2z_2} + \frac{2z_2}{z_1} = i$ and if 0, z_1, z_2 form two non similar triangles and if α, β are the least angles in the two triangles, then $\cot\alpha + \cot\beta$ equals

(a) $\sqrt{5}$ (b) $2\sqrt{5}$ (c) 1 (d) 2

11. $\left(\frac{a}{b} - \frac{b}{a}\right)\tan\left(i \ln\left(\frac{a - ib}{a + ib}\right)\right) =$ ___ where $a, b \in R^+$ and $i = \sqrt{-1}$.

(a) 1 (b) 2 (c) 1/2 (d) -1

Multiple Correct Answer Type

12. Consider two curves represented by

$$\arg(z - z_1) = \frac{3\pi}{4} \text{ and } \arg(2z + 1 - 2i) = \frac{\pi}{4}$$

(a) Two curves do not intersect if $z_1 = 3i$
(b) Two curves do not intersect if $z_1 = 2 + i$
(c) Two curves intersect if $z_1 = 3 + i$
(d) Two curves intersect at $\frac{3}{4}+i\frac{9}{4}$ if $z_1 = 3$

13. For complex numbers z and ω, if $|z|^2\omega - |\omega|^2 z = z - \omega$ and $z \neq \omega$, then
(a) $z=\bar{\omega}$ (b) $z=-\omega$ (c) $z\bar{\omega}=1$ (d) $\bar{z}\omega=1$

14. If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1|=|z_2|=1$ and $\text{Re}(z_1\bar{z}_2)=0$, then the pair of complex numbers $\omega_1 = a + ic$ and $\omega_2 = b + id$ satisfies
(a) $|\omega_1| = 1$ (b) $|\omega_2| = 1$
(c) $\text{Re}(\omega_1\bar{\omega}_2)=0$ (d) $\omega_1\bar{\omega}_2=0$

15. If $z_1 = 5 + 12i$ and $|z_2| = 4$, then
(a) maximum $(|z_1 + iz_2|) = 17$
(b) minimum $(|z_1 + (1 + i)z_2|) = 13-9\sqrt{2}$
(c) minimum $\left|\frac{z_1}{z_2+\frac{4}{z_2}}\right|=\frac{13}{4}$
(d) maximum $\left|\frac{z_1}{z_2+\frac{4}{z_2}}\right|=\frac{13}{3}$

16. A, B, C are the points representing the complex numbers z_1, z_2, z_3 respectively on the complex plane and the circumcentre of the triangle ABC lies at the origin. If the altitude AD of the triangle ABC meets the circumcircle again at P, then P represents the complex number
(a) $-\bar{z}_1z_2z_3$ (b) $-\frac{\bar{z}_1z_2}{z_3}$
(c) $-\frac{\bar{z}_1z_3}{z_2}$ (d) $-\frac{z_2z_3}{z_1}$

17. If points A and B are represented by the non-zero complex numbers z_1 and z_2 on the Argand plane such that $|z_1 + z_2| = |z_1 - z_2|$ and O is the origin, then
(a) orthocentre of ΔOAB lies at O
(b) circumcentre of ΔAOB is $\frac{z_1+z_2}{2}$
(c) $\arg\left(\frac{z_1}{z_2}\right)=\pm\frac{\pi}{2}$
(d) ΔOAB is isosceles

18. If the lines $a\bar{z}+\bar{a}z+b=0$ and $c\bar{z}+\bar{c}z+d=0$ are mutually perpendicular, where a and c are non-zero complex numbers and b and d are real numbers, then
(a) $a\bar{a}+c\bar{c}=0$ (b) $a\bar{c}$ is purely imaginary
(c) $\arg\left(\frac{a}{c}\right)=\pm\frac{\pi}{2}$ (d) $\frac{a}{\bar{a}}=\frac{c}{\bar{c}}$

19. Let z be a complex number satisfying $|z + 1| < |z - 2|$. If $\omega = 3z + 2 + i$, then
(a) $|\omega + 1| < |\omega - 8|$ (b) $|\omega + 1 + i| < |\omega - 8 + i|$
(c) $|\omega + 5| < |\omega - 4|$ (d) $|\omega - 12 + i| < |\omega - 3 + i|$

20. $\sum_{k=1}^{6}\left(\sin\frac{2k\pi}{7}-i\cos\frac{2k\pi}{7}\right)=$
(a) -1 (b) 0 (c) $-i$ (d) i

21. If from a point $P(z_1)$ on the curve $|z| = 2$, pair of tangents are drawn to the curve $|z| = 1$ meeting at $Q(z_2)$, $R(z_3)$, then
(a) complex number $\frac{z_1+z_2+z_3}{3}$ will lie on the curve $|z| = 1$
(b) $\left(\frac{4}{\bar{z}_1}+\frac{1}{\bar{z}_2}+\frac{1}{\bar{z}_3}\right)\left(\frac{4}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right)=9$
(c) orthocentre and circumcentre of ΔPQR will coincide
(d) $(z_1+z_2+z_3)(\bar{z}_1+\bar{z}_2+\bar{z}_3)=9$

22. Let z_1, z_2, z_3 be the vertices of a triangle ABC. Then which of the following statements is/are correct?
(a) If $\frac{1}{z-z_1}+\frac{1}{z-z_2}+\frac{1}{z-z_3}=0$, where $z=\frac{z_1+z_2+z_3}{3}$, then ABC is an equilateral triangle.
(b) If ABC is an equilateral triangle, then $\frac{1}{z-z_1}+\frac{1}{z-z_2}+\frac{1}{z-z_3}=0$, where $z=\frac{z_1+z_2+z_3}{3}$
(c) If $\begin{vmatrix}1 & 1 & 1\\ z_1 & z_2 & z_3\\ z_2 & z_3 & z_1\end{vmatrix}=0$, then the triangle ABC is equilateral.
(d) If $|z_1| = |z_2| = |z_3|$ and $z_1 + z_2 + z_3 = 0$, then the triangle ABC is equilateral.

Comprehension Type

Paragraph for Q. No. 23 to 25

Suppose z_1, z_2 and z_3 represent the vertices A, B and C of an equilateral triangle ABC on the Argand plane, then $|z_3 - z_1| = |z_2 - z_1| = |z_3 - z_2|$ or $z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = 0$

23. If the complex numbers z_1, z_2, z_3 represent the vertices of an equilateral triangle such that $|z_1| = |z_2| = |z_3|$, then $z_1 + z_2 + z_3$ is
(a) 0 (b) ω (c) ω^2 (d) 3

24. The roots z_1, z_2, z_3 of the equation $x^3 + 3px^2 + 3qx + r = 0$, $(p, q, r \in C)$ form an equilateral triangle in the Argand plane if and only if

(a) $p^2 = q$ (b) $p = q^2$
(c) $p = q$ (d) $|p| = |q|$

25. If $|z| = 2$, the area of the triangle whose sides are $|z|$, $|\omega z|$ and $|z + \omega z|$ (where ω is a complex cube root of unity) is

(a) $2\sqrt{3}$ (b) $\frac{3\sqrt{3}}{2}$
(c) 1 (d) $\sqrt{3}$

Paragraph for Q. No. 26 to 28

Let z be a complex number and K be a real number. Consider the sets

$A : \{z : |\text{Im}(z)| = K - |\text{Re}(z) - K|\}$,

$B : \{z : |z - K| > |z - 2K|\}$ and

C : Circle inscribed in the geometrical figure formed by A

26. Area bounded by A is

(a) $\frac{3K^2}{2}$ (b) K^2 (c) $2K^2$ (d) $\frac{K^2}{2}$

27. Radius of C is

(a) $\frac{K}{2}$ (b) $\frac{K}{\sqrt{2}}$ (c) K (d) $\frac{3K}{2}$

28. Number of points of contact of C with A that belong to B is

(a) 0 (b) 2 (c) 3 (d) 4

Matrix-Match Type

29. z_1, z_2, z_3 are the vertices of a triangle.
Match the triangles and the conditions given in the following columns :

	Column-I		Column-II
(A)	$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_1z_3$	(p)	Right angled
(B)	$\text{Re}\frac{z_3 - z_1}{z_3 - z_2} = 0$	(q)	Obtuse angled
(C)	$\text{Re}\frac{z_3 - z_1}{z_3 - z_2} < 0$	(r)	Isosceles and right angled
(D)	$\frac{z_3 - z_1}{z_3 - z_2} = i$	(s)	Equilateral

30. Match the following :

	Column-I		Column-II
(A)	If G is the greatest and L is the least value of $z + 2 + i$, and $\|z + 2 + i\| \leq 1$ where $i = \sqrt{-1}$, then	(p)	$LG = 9$
(B)	If G is the greatest and L is the least value of $\|z + 2i\|$, where $i = \sqrt{-1}$, and $1 \leq \|z - 1\| \leq 3$, then	(q)	$L + G = 8$
(C)	If G is the greatest and L is the least value of $\|z - 2\|$ and $\|z + i\| \leq 1$ where $i = \sqrt{-1}$, then	(r)	$(\sqrt{2G} - \sqrt{2L})^2 = 4$
		(s)	$LG = 4$

Integer Answer Type

31. If $z_1, z_2, z_3 \in C$ satisfy the system of equations given by $|z_1| = |z_2| = |z_3| = 1$, $z_1 + z_2 + z_3 = 1$ and $z_1z_2z_3 = 1$ such that $\text{Im}(z_1) < \text{Im}(z_2) < \text{Im}(z_3)$, then the value of $[|z_1 + z_2^2 + z_3^3|]$is, (where $[\cdot]$ denotes the greatest integer function).

32. If the complex numbers z for which $\arg\left[\frac{3z-6-3i}{2z-8-6i}\right] = \frac{\pi}{4}$ and $|z - 3 + i| = 3$, are $\left(k - \frac{4}{\sqrt{5}}\right) + i\left(1 + \frac{2}{\sqrt{5}}\right)$ and $\left(k + \frac{4}{\sqrt{5}}\right) + i\left(1 - \frac{2}{\sqrt{5}}\right)$, then k must be equal to

33. The maximum value of $|z|$ when z satisfies the condition $\left|z + \frac{2}{z}\right| = 2$ is $\sqrt{\lambda} + 1$. Find the value of λ.

34. If $|z_1 - z_2| = \sqrt{25 - 12\sqrt{3}}$, and $\frac{z_1 - z_3}{z_2 - z_3} = \frac{3}{4}e^{i\frac{\pi}{6}}$ then area of triangle (in square units) whose vertices are represented by z_1, z_2, z_3 is

35. Let z_1, z_2 be the roots of the equation $z^2 + az + b = 0$ where a and b may be complex. Let A and B represent z_1 and z_2 in the Argand's plane. If $\angle AOB = \alpha \neq 0$ and $OA = OB$. Then $\alpha^2 = \lambda b \cos^2\left(\frac{\alpha}{2}\right)$, where value of λ is

36. If $|z_1| = 2$, $|z_2| = 3$, $|z_3| = 4$ and $|2z_1 + 3z_2 + 4z_3| = 4$ then $\frac{|8z_2z_3 + 27z_3z_1 + 64z_1z_2|}{16} =$ ________.
(where z_1, z_2, z_3 are complex numbers)

37. If $z+\frac{1}{z}=1$ and $a=z^{2005}+\frac{1}{z^{2005}}$ and b is last digit of the number $2^{2^n}-1$ when the integer $n>1$ and $a^2+b^2=13\lambda$, then λ = _______

38. For all complex number z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$ the minimum value of $|z_1 - z_2|$ is _______.

39. If z_1 lies on the circle $|z| = 3$ and $x+iy=z_1+\frac{1}{z_1}$ then $\frac{x^2}{100}+\frac{y^2}{64}=\frac{1}{k}$, where k is equal to _______

40. z_1, z_2 are roots of the equation $z^2 + az + b = 0$. If ΔOAB(O is origin), A and B represent z_1 and z_2 is equilateral Δ. If $a^2/\lambda b$ satisfies it, then λ = _____.

SOLUTIONS

1. (b) : Let the equation of required circle

$|z - (a + i)| = r$ ...(1)

Where 'r' is radius of required circle

Given circles are $|z| = 1$ ...(2)

$|z - 1| = 4$ ...(3)

Circles in (1) & (2) intersects orthogonally

$\Rightarrow$ Square of distance between centres = Sum of squares of radii

$\Rightarrow |a + i - 0|^2 = r^2 + 1$

$a^2 + 1 = r^2 + 1 \Rightarrow |a| = r$

Now, circles in (1) & (3) intersects orthogonally

$|a + i - 1|^2 = r^2 + 16 = (a - 1)^2 + 1 = r^2 + 16$

But $r = |a|$ $\therefore$ $(a - 1)^2 + 1 = a^2 + 16$

$\Rightarrow -2a = 14 \Rightarrow a = -7$

$\therefore$ $r = 7$

$\therefore$ Required circle is $|z - (-7 + i)| = 7 \Rightarrow |z + 7 - i| = 7$

2. (a) : 1, α_1, α_2, ..., α_{n-1} are n^{th} roots of unity

These are the roots of $x^n - 1 = 0$

Let $y=\frac{1}{1-\alpha}$ where $\alpha=\alpha_1,\alpha_2,......,\alpha_{n-1}$

$1-\alpha=\frac{1}{y}\Rightarrow\alpha=\frac{y-1}{y}$

But α is a root of $x^n - 1 = 0$ $\therefore$ $\alpha^n = 1$ $\Rightarrow$ $(y - 1)^n = y^n$

$\Rightarrow y^n - {}^nC_1\cdot y^{n-1} + {}^nC_2 y^{n-2} - ...... + (-1)^n = y^n$

$\Rightarrow -{}^nC_1\cdot y^{n-1} + {}^nC_2\cdot y^{n-2} - ...... + (-1)^n = 0$

Sum of roots

$$\frac{1}{1-\alpha_1}+\frac{1}{1-\alpha_2}+........+\frac{1}{1-\alpha_{n-1}}=\frac{{}^nC_2}{{}^nC_1}=\frac{n-1}{2}$$

3. (b) : $\bar{z}=-z$ and $z\bar{z}-4zi=z^2$

$\Rightarrow -z^2 - 4zi = z^2$

$\Rightarrow 2z^2 + 4zi = 0$

$\Rightarrow 2z(z + 2i) = 0$

$\Rightarrow z = -2i \Rightarrow |z| = 2$

4. (d) : $|z\omega| = 1 \Rightarrow |z||\omega| = 1$

$\therefore$ $|\omega|=\frac{1}{|z|}$ ...(i)

Let $z = re^{i\theta}, \bar{z} = re^{-i\theta}$

Given Arg ω = Arg $z - \pi/2 = \theta - \pi/2$

$\therefore$ $\omega=\frac{1}{r}e^{i(\theta-\pi/2)}$ by (i)

$\therefore$ $\bar{z}\omega=(re^{-i\theta}).\frac{1}{r}e^{i(\theta-\pi/2)}=e^{-i\pi/2}$

$=\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}=-i$

5. (d) : If $\omega=re^{i\theta}$ then $\frac{1}{\omega}=\frac{1}{r}e^{-i\theta}$

$\therefore$ $z=\left(\omega+\frac{1}{\omega}\right)$

$=r(\cos\theta+i\sin\theta)+\frac{1}{r}(\cos\theta-i\sin\theta)$

$\therefore$ $x=\left(r+\frac{1}{r}\right)\cos\theta,\ y=\left(r-\frac{1}{r}\right)\sin\theta$

Eliminating θ, $\frac{x^2}{\left(r+\frac{1}{r}\right)^2}+\frac{y^2}{\left(r-\frac{1}{r}\right)^2}=1$

Above represents an ellipse

$\therefore$ distance between foci is

$$2ae=2\sqrt{a^2\left(1-\frac{b^2}{a^2}\right)}=2\sqrt{a^2-b^2}=2\sqrt{4}=4$$

6. (d) : The two equations represent circles

$x^2 + y^2 = 9$ and $(x + 1)^2 + (y - 1)^2 = 2$

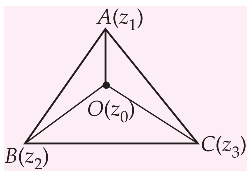
$C_1(0,0), r_1 = 3$ and $C_2(-1,1), r_2=\sqrt{2}$

$C_1C_2=\sqrt{2}$ and $r_1-r_2=3-\sqrt{2}$

Now $\sqrt{2}<3-\sqrt{2}$ as $2\sqrt{2}<3 \Rightarrow 8<9$.

$\therefore$ $C_1C_2 < r_1 - r_2$. Hence the two circles do not intersect but one lies completely within the other. Hence there is no solution.

7. (c) : Taking rotation at O



$\frac{z_0-z_1}{z_0-z_2}=\cos 2C-i\sin 2C$

$\frac{z_0-z_3}{z_0-z_2}=\cos 2A+i\sin 2A$

$$\text{Now}\left(\frac{z_0-z_1}{z_0-z_2}\right)\frac{\sin 2A}{\sin 2B}+\left(\frac{z_0-z_3}{z_0-z_2}\right)\frac{\sin 2C}{\sin 2B}$$

$$=\frac{\sin 2A\cos 2C - i\sin 2A\sin 2C + \cos 2A\sin 2C + i\sin 2A\sin 2C}{\sin 2B}$$

$$=\frac{\sin(2A+2C)}{\sin 2B}=-1$$

8. (d) : Given $z = x + iy$, then

$|z - 4| < |z - 2| \;\Rightarrow\; -8x + 16 < -4x + 4$

$\Rightarrow\; 4x > 12 \;\Rightarrow\; x > 3$

9. (b) : $z_1(z_1^2 - 3z_2^2) = 2$ ...(1)

$z_2(3z_1^2 - z_2^2) = 11$ ...(2)

On multiplying (2) by i and adding to (1), we get

$(z_1 + iz_2)^3 = 2 + i11$ ...(3)

and on multiplying (2) by i and subtracting from (1), we get

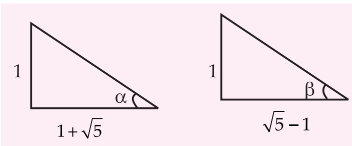
$(z_1 - iz_2)^3 = 2 - 11i$ ... (4)

On multiplying (3) and (4), we get $z_1^2 + z_2^2 = 5$

10. (b) : $\dfrac{z_1}{2z_2} = t \;\Rightarrow\; t + \dfrac{1}{t} = i \;\Rightarrow\; t^2 - it + 1 = 0$

$$\Rightarrow\; t = \frac{i \pm \sqrt{-1-4}}{2} = \frac{(1+\sqrt{5})i}{2}, \frac{(\sqrt{5}-1)(-i)}{2}$$

$$\therefore\; \frac{z_1}{z_2} = (1+\sqrt{5})i \;\text{ or }\; \frac{z_1}{z_2} = (\sqrt{5}-1)(-i)$$



$\therefore\; \cot\alpha = 1 + \sqrt{5}$ and $\cot\beta = \sqrt{5} - 1$

Now, $\cot\alpha + \cot\beta = 1 + \sqrt{5} + \sqrt{5} - 1 = 2\sqrt{5}$

11. (b) : $\dfrac{a^2-b^2}{ab}\tan\left\{i(\ln(a-ib) - \ln(a+ib))\right\}$

$$= \frac{a^2-b^2}{ab}\tan\left\{i(\ln re^{-i\theta} - \ln re^{i\theta})\right\}, \text{ where } \theta = \tan^{-1}\frac{b}{a}$$

and $r = \sqrt{a^2+b^2}$

$$= \frac{a^2-b^2}{ab}\tan\{-2i \times i\theta\} = \frac{a^2-b^2}{ab} \times \tan 2\theta$$

$$= \frac{a^2-b^2}{ab} \times \frac{2ab}{a^2-b^2} = 2$$

12. (a, c, d)

13. (c, d) : $|z|^2\omega - |\omega|^2 z = z - \omega$

$\Rightarrow\; (|z|^2 + 1)\omega = (1 + |\omega|^2)z$

$\therefore\; \dfrac{z}{\omega} = \dfrac{|z|^2+1}{|\omega|^2+1}$ is purely real

$\therefore\; \dfrac{z}{\omega} = \dfrac{\bar{z}}{\bar{\omega}} \;\Rightarrow\; z\bar{\omega} = \bar{z}\omega$ ...(i)

Now, $|z|^2\omega - |\omega|^2 z = z - \omega \;\Rightarrow\; z\bar{z}\omega - \omega\bar{\omega}z = z - \omega$

$\Rightarrow\; (\bar{z}\omega - 1)(z - \omega) = 0$ [Using (i)]

$\Rightarrow\; z = \omega$ or $\bar{z}\omega = 1$

Similarly, $z = \omega$ or $z\bar{\omega} = 1$

14. (a, b, c) : $|z_1| = |z_2| = 1 \Rightarrow a^2 + b^2 = c^2 + d^2 = 1$ ...(1)

and $\text{Re}(z_1\bar{z}_2) = 0 \;\Rightarrow\; \text{Re}\{(a + ib)(c - id)\} = 0$

$\Rightarrow ac + bd = 0$ ...(2)

Now from (1) & (2), $a^2 + b^2 = 1$

$\Rightarrow\; a^2 + \dfrac{a^2c^2}{d^2} = 1 \Rightarrow a^2 = d^2$ ...(3)

Also $c^2 + d^2 = 1 \;\Rightarrow\; c^2 + \dfrac{a^2c^2}{b^2} = 1 \Rightarrow b^2 = c^2$ ...(4)

$|\omega_1| = \sqrt{a^2+c^2} = \sqrt{a^2+b^2} = 1$ [from (1) & (4)]

and $|\omega_2| = \sqrt{b^2+d^2} = \sqrt{b^2+a^2} = 1$ [from (1) & (3)]

Further

$\text{Re}(\omega_1\bar{\omega}_2) = \text{Re}\{(a+ic)(b-id)\} = ab + cd = 0$

Also, $\text{Im}(\omega_1\bar{\omega}_2) = bc - ad = \pm 1$

$\Rightarrow\; |\omega_1| = 1, |\omega_2| = 1$ and $\text{Re}(\omega_1\bar{\omega}_2) = 0$

15. (a, d) : $z_1 = 5 + 12i$, $|z_2| = 4$

$|z_1 + iz_2| \le |z_1| + |z_2| = 13 + 4 = 17$

$\therefore\; |z_1 + (1+i)z_2| \ge ||z_1| - |1+i||z_2|| = 13 - 4\sqrt{2}$

$\therefore\; \min(|z_1 + (1 + i)z_2|) = 13 - 4\sqrt{2}$

$$\left|z_2 + \frac{4}{z_2}\right| \le |z_2| + \frac{4}{|z_2|} = 4 + 1 = 5$$

$$\left|z_2 + \frac{4}{z_2}\right| \ge |z_2| - \frac{4}{|z_2|} = 4 - 1 = 3$$

$$\therefore\; \max\left|\frac{z_1}{z_2 + \frac{4}{z_2}}\right| = \frac{13}{3} \text{ and } \min\left|\frac{z_1}{z_2 + \frac{4}{z_2}}\right| = \frac{13}{5}$$

16. (b, c, d) :

We have $\angle DAC = \dfrac{\pi}{2} - C$ and $OC = OD$

$$\therefore\; \frac{z}{z_3} = \cos(\pi - 2C) + i\sin(\pi - 2C)$$

or $\frac{z}{z_3} = -\cos 2C + i\sin 2C$ ...(i)

Again $\angle AOB = 2C$ and $OA = OB$

$\therefore \quad \frac{z_1}{z_2} = \cos 2C + i\sin 2C$ ...(ii)

Multiply (i) and (ii)

$\Rightarrow \frac{zz_1}{z_2 z_3} = -1 \Rightarrow z = -\frac{z_2 z_3}{z_1} = -\frac{\bar{z}_1\bar{z}_2 z_2 z_3}{z_1\bar{z}_1\bar{z}_2} = -\frac{\bar{z}_1 z_3}{\bar{z}_2}$

$(\because z_1\bar{z}_1 = z_2\bar{z}_2)$

$= -\frac{\bar{z}_1 z_2}{\bar{z}_3}$

17. (a, b, c) : $|z_1 + z_2| = |z_1 - z_2|$

$\Rightarrow (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$\Rightarrow z_1\bar{z}_2 + z_2\bar{z}_1 = 0$ ...(i)

$\Rightarrow \frac{z_1}{z_2} = -\left(\frac{\bar{z}_1}{\bar{z}_2}\right) \Rightarrow \frac{z_1}{z_2}$ is purely imaginary

$\Rightarrow \arg\left(\frac{z_1}{z_2}\right) = \pm\frac{\pi}{2}$

Also from (i) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2$

$\Rightarrow$ ΔAOB is a right angle triangle, right angled at O.

So, circumcentre $= \frac{z_1 + z_2}{2}$.

18. (b, c) : Let $a = a_1 + ia_2$ and $c = c_1 + ic_2$, then

Slope of the line $a\bar{z} + \bar{a}z + b = 0$ is $-\frac{a_1}{a_2}$ and slope of the line $c\bar{z} + \bar{c}z + d = 0$ is $-\frac{c_1}{c_2}$

So, $-\frac{a_1}{a_2} \times \left(-\frac{c_1}{c_2}\right) = -1 \Rightarrow a_1c_1 + a_2c_2 = 0$

$\Rightarrow \left(\frac{a+\bar{a}}{2}\right)\left(\frac{c+\bar{c}}{2}\right) + \left(\frac{a-\bar{a}}{2i}\right)\left(\frac{c-\bar{c}}{2i}\right) = 0 \Rightarrow a\bar{c} + \bar{a}c = 0$

$\therefore \quad a\bar{c}$ is purely imaginary. Also $\frac{a}{c} = -\frac{\bar{a}}{\bar{c}} \Rightarrow \frac{a}{c}$ is also purely imaginary.

$\Rightarrow \arg\left(\frac{a}{c}\right) = \pm\frac{\pi}{2}$

19. (a, b, c) : $|z+1| < |z-2| \Rightarrow z + \bar{z} < 1$

So, $\omega + \bar{\omega} = 3(z + \bar{z}) + 4 \Rightarrow \omega + \bar{\omega} < 7$

Now, $|\omega + 1| < |\omega - 8|$ if $\omega + \bar{\omega} < 7$, which is true

$|\omega + 1 + i| < |\omega - 8 + i|$ if $\omega + \bar{\omega} < 7$ which is true

$|\omega + 5| < |\omega - 4|$ if $\omega + \bar{\omega} < -1$, which is true

$|\omega - 12 + i| < |\omega - 3 + i|$ if $\omega + \bar{\omega} > 15$,

which is not true

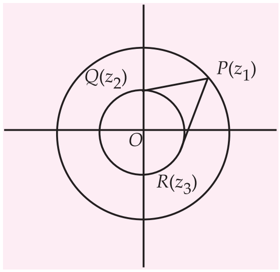
20. (d) : $\sum_{k=1}^{6} -i\left(\cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7}\right) = -i\sum_{k=1}^{6} cis\left(\frac{2k\pi}{7}\right)$,

$= -i(\alpha + \alpha^2 + \alpha^3 + \ldots.. + \alpha^6)$ $\left[\text{Put } cis\left(\frac{2k\pi}{7}\right) = \alpha\right]$

$= -i(-1)$ $[\because 1 + \alpha + \alpha^2 + \ldots + \alpha^6 = 0]$

$= i$

21. (a, b, c, d) : As ΔPQR is equilateral orthocentre, circumcentre, centroid will coincide



$\left|\frac{z_1 + z_2 + z_3}{3}\right| = 1$

$\Rightarrow |z_1 + z_2 + z_3|^2 = 9$

$\Rightarrow (z_1 + z_2 + z_3)(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) = 9$

$\Rightarrow \left(\frac{4}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3}\right)\left(\frac{4}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right) = 9$

$(\because |z_1|^2 = 4, |z_2|^2 = 1, |z_1|^2 = 1)$

22. (a, b, c, d) : A necessary and sufficient condition for a triangle having vertices z_1, z_2 and z_3 to be an equilateral triangle is

$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_1z_3 + z_2z_3$.

(a) and (b) will follow by performing some algebraic jugglery on the known condition given above.

To prove (d) note that $z_1 + z_2 + z_3 = 0$ can be changed to $\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}=0 \quad (\because |z_1|=|z_2|=|z_3|)$

23. (a) : $\frac{z_1+z_2+z_3}{3}=0$

$z_1 + z_2 + z_3 = 0$

24. (a) : $\Sigma z_1 = -3p, \Sigma z_1 z_2 = 3q, z_1 z_2 z_3 = -r$

$(z_1 + z_2 + z_3)^2 = 3(z_1z_2 + z_2z_3 + z_3z_1)$

$\Rightarrow 9p^2 = 9q \Rightarrow p^2 = q$

25. (d) : We have $|z| = 2$, $|\omega z| = |\omega||z|$ and $|z + \omega z| = |-\omega^2 z| = 2$

$\therefore$ Area of triangle $=\frac{\sqrt{3}}{4}\cdot 2^2=\sqrt{3}$

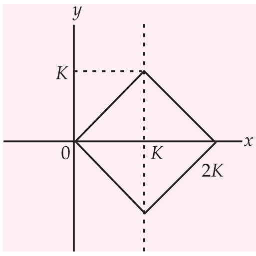
26. (c) : $z = x + iy$

$|y| = K - |x - K|$

$\Rightarrow K \geq |x - K|$

$\Rightarrow 0 \leq x \leq 2K.$

Area $=4\cdot\frac{1}{2}K\cdot K=2K^2$



27. (b) : Since the curve is symmetrical about $y = 0$; $x = K$ the centre of the circle $(K, 0)$ and it touches $y = |x|$.

$\Rightarrow$ Radius $=\frac{K}{\sqrt{2}}$

28. (a) : B is $|z - K| > |z - 2K|$ is the region to the right of $x=\frac{K+2K}{2}=\frac{3K}{2}$

Points of contact of the circle $(x-K)^2+y^2=\frac{K^2}{2}$ with A are $\left(\frac{K}{2},\pm\frac{K}{2}\right)\left(\frac{3K}{2},\pm\frac{K}{2}\right)$

Same $x>\frac{3K}{2}$ for B we have no points

29. (A) → (s); (B) → (p); (C) → (q); (D) → (r)

(A) z_1, z_2, z_3 are the vertices of an equilateral triangle if $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$.

(B) Since, $\frac{z_3-z_1}{z_3-z_2}$ is purely imaginary, the triangle is right angled with right angle at z_3.

(C) It is a obtuse angled triangle.

(D) $\frac{z_3-z_1}{z_3-z_2}$ is purely imaginary, the triangle is right angled and isosceles

30. (A) → (p); (B) → (r, s); (C) → (s)

31. (2) : We have,

$$z_1z_2+z_2z_3+z_1z_2z_3\left(\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right)=1.$$

$$\left(\frac{|z_1|^2}{z_1}+\frac{|z_2|^2}{z_2}+\frac{|z_3|^2}{z_3}\right)=\bar{z}_1+\bar{z}_2+\bar{z}_3=\left(\overline{z_1+z_2+z_3}\right)=1$$

$\therefore$ The cubic equation with roots z_1, z_2 and z_3 in z will be $(z-z_1)(z-z_2)(z-z_3)=z^3-1\cdot z^2+1\cdot z-1=0$

$\Rightarrow (z - 1)(z^2 + 1) = 0, z = 1, \pm i$

$\because \operatorname{Im}(z_1) < \operatorname{Im}(z_2) < \operatorname{Im}(z_3)$

$\therefore z_1 = -i, z_2 = 1, z_3 = i$

Now $\left|z_1+z_2^2+z_3^3\right|=\left|-i+1^2+i^3\right|=|1-2i|=\sqrt{5}$

Hence $\left[\left|z_1+z_2^2+z_3^3\right|\right]=\left[\sqrt{5}\right]=2$

32. (4) : The first relation can be written as

$$\operatorname{Arg}\frac{3}{2}+\operatorname{Arg}\frac{z-2-i}{z-4-3i}=\frac{\pi}{4}$$

$$\Rightarrow \operatorname{Arg}\frac{z-2-i}{z-4-3i}=\frac{\pi}{4} \quad \left(\because \operatorname{Arg}\frac{3}{2}=0\right)$$

$$\operatorname{Arg}\frac{x+iy-2-i}{x+iy-4-3i}=\frac{\pi}{4} \Rightarrow \operatorname{Arg}\frac{(x-2)+i(y-1)}{(x-4)+i(y-3)}=\frac{\pi}{4}$$

$$\Rightarrow \operatorname{Arg}\frac{[(x-2)+i(y-1)][(x-4)-i(y-3)]}{(x-4)^2+(y-3)^2}=\frac{\pi}{4}$$

$$\Rightarrow \operatorname{Arg}\frac{\begin{bmatrix}(x-2)(x-4)+(y-1)(y-3)\\+i(y-1)(x-4)-i(x-2)(y-3)\end{bmatrix}}{(x-4)^2+(y-3)^2}=\frac{\pi}{4}$$

$$=\tan^{-1}\frac{(y-1)(x-4)-(x-2)(y-3)}{(x-2)(x-4)+(y-1)(y-3)}=\frac{\pi}{4}$$

The other equation is also a circle given by

$x^2 + y^2 - 6x + 2y + 1 = 0$

The two circles intersect at

$$\left(4-\frac{4}{\sqrt{5}},1+\frac{2}{\sqrt{5}}\right)\text{and}\left(4+\frac{4}{\sqrt{5}},1-\frac{2}{\sqrt{5}}\right)\Rightarrow K=4$$

33. (3) : $|z|=\left|z-\frac{2}{z}+\frac{2}{z}\right|\leq\left|z-\frac{2}{z}\right|+\frac{2}{|z|}=2+\frac{2}{|z|}$

$\Rightarrow |z|^2 - 2|z| - 2 \leq 0$

Solving the quadratic, we get $|z|\leq 1+\sqrt{3}\Rightarrow\lambda=3$

34. (3) : $\left|\frac{z_1-z_3}{z_2-z_3}\right|=\frac{3}{4}$

Let $|z_1-z_3|=3k, |z_2-z_3|=4k$

Angle at $z_3 = \frac{\pi}{6}$

$$\cos 30° = \frac{16k^2 + 9k^2 - 25 + 12\sqrt{3}}{2 \times 4k \times 3k} \Rightarrow k = 1$$

$$\text{Area} = \frac{1}{2} \cdot 3 \cdot 4 \cdot \sin 30° = 3 \text{ sq. units}$$

35. (4) : $z_1 + z_2 = -a,\ z_1 z_2 = b$

$\therefore\ \frac{z_2}{z_1} = \cos\alpha + i\sin\alpha$

$\therefore\ (z_2 - z_1)^2 = -4\sin^2\frac{\alpha}{2}(\cos\alpha + i\sin\alpha)$

$\therefore\ (z_1 + z_2)^2 - 4z_1 z_2 = -4z_1 z_2 \sin^2\frac{\alpha}{2}$

$a^2 = 4b\cos^2\frac{\alpha}{2}$

36. (6) : $|8z_2 z_3 + 27z_3 z_1 + 64z_1 z_2|$

$$= |z_1||z_2||z_3|\left|\frac{8}{z_1} + \frac{27}{z_2} + \frac{64}{z_3}\right| = 24\left|\frac{8\bar{z}_1}{4} + \frac{27\bar{z}_2}{9} + \frac{64\bar{z}_3}{16}\right|$$

$$= 24|2z_1 + 3z_2 + 4z_3| = 96$$

37. (2) : $z + \frac{1}{z} = 1 \quad \Rightarrow z = -\omega, -\omega^2$

$z^{2005} + \frac{1}{z^{2005}} = -\omega - \omega^2 = 1$ for $z = -\omega$ or $-\omega^2$

$\therefore\ a = 1$

If $n > 1,\ 2^{2^n} = 2^{4k},\ k \in Z$ i.e. 16^k

16^k has last digit 6

$\therefore\ b = 6 - 1 = 5$

$\therefore\ \frac{a^2 + b^2}{13} = 2$

38. (2) : $C_1(0,0)$ is centre of bigger circle and $C_2(3,4)$ is centre of smaller circle

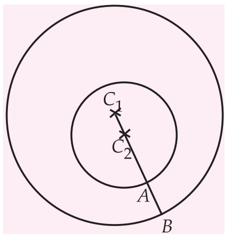
$C_1B = r_1 = 12$ (radius of bigger circle)

$C_2A = r_2 = 5$ (radius of smaller circle)

$C_1C_2 = 5$ minimum value of $|z_1 - z_2| = AB$

$\therefore\ C_1C_2 + C_2 + AB = C_1B$

$\therefore\ AB = 12 - 5 - 5 = 2$



39. (9) : $x + iy = 3 \text{ cis } \theta + \frac{1}{3}\text{cis }(-\theta)$

$x = \frac{10}{3}\cos\theta,\ y = \frac{8}{3}\sin\theta$

$\Rightarrow \frac{9x^2}{100} + \frac{9y^2}{64} = 1 \quad \Rightarrow \frac{x^2}{100} + \frac{y^2}{64} = \frac{1}{9}$

$\therefore\ k = 9$

40. (3) : z_1, z_2, z_3 form equilateral triangle if and only if $z_1^2 + z_2^2 + z_3^2 = \Sigma z_1 z_2$.

Take $z_3 = 0$

$\therefore\ z_1^2 + z_2^2 = z_1 z_2 \Rightarrow (z_1 + z_2)^2 = 3z_1 z_2$

$\Rightarrow a^2 = 3b$

BRAIN @ WORK

BINOMIAL THEOREM

This article is a collection of shortcut methods, important formulas and MCQs along with their detailed solutions which provides an extra edge to the readers who are preparing for various competitive exams like JEE(Main & Advanced) and other PETs.

MEANING OF BINOMIAL

In Earlier classes, we have learnt how to find the squares and cubes of binomials like $a + b$ and $a - b$. However, for higher powers like $(a + b)^5$, the calculations become difficult by using repeated multiplication. The difficulty was overcome by a theorem known as binomial theorem. An algebraic expression consisting of two terms with positive or negative sign between them is called a binomial expression. For example,

$$(a + b),\ (2x - 3y),\ \left(\frac{p}{x^2}+\frac{q}{x^4}\right),\left(\frac{1}{x}+\frac{4}{y^3}\right) \text{ etc.}$$

Similarly, an algebraic expression containing three terms is called a trinomial expression. In general, expressions containing more than two terms are known as multinomial expression. The general form of the binomial is $(x + a)$ and the expansion of $(x + a)^n$, $n \in N$ is called the binomial theorem.

BINOMIAL THEOREM

The formula by which any power of a binomial expression can be expanded in the form of a series is known as binomial theorem. This theorem was given by Sir Issac Newton.

BINOMIAL THEOREM FOR POSITIVE INTEGRAL INDEX

If n is a positive integer and $x, y \in C$ then

$$(x + y)^n = {}^nC_0\, x^{n-0} y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \dots + {}^nC_r\, x^{n-r} y^r + \dots + {}^nC_n\, x^0 y^n \quad \dots \text{(i)}$$

$$= \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r$$

Here ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ..., ${}^nC_n$ are called binomial coefficients.

In particular

1. Replacing y by $-y$ in (i), we get
$$(x - y)^n = {}^nC_0 x^{n-0} y^0 - {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 - \dots + {}^nC_r(-1)^r x^{n-r} y^r + \dots + (-1)^n\, {}^nC_n\, x^0 y^n \quad \dots \text{(ii)}$$
$$= \sum_{r=0}^{n} (-1)^r\, {}^nC_r x^{n-r} y^r$$
2. Adding (i) and (ii), we get
$$(x + y)^n + (x - y)^n = 2\,\{x^n + {}^nC_2\, x^{n-2} y^2 + {}^nC_4\, x^{n-4} y^4 + \dots\dots\}$$
$$= 2\,\{\text{sum of terms at odd places}\}$$
The last term is ${}^nC_n\, y^n$ or ${}^nC_{n-1}\, xy^{n-1}$ according as n is even or odd respectively.
3. Subtracting (ii) from (i), we get
$$(x + y)^n - (x - y)^n = 2\{{}^nC_1\, x^{n-1} y^1 + {}^nC_3\, x^{n-3} y^3 + \dots\dots\}$$
$$= 2\,\{\text{sum of terms at even places}\}$$
The last term is ${}^nC_{n-1}\, xy^{n-1}$ or ${}^nC_n\, y^n$ according as n is even or odd respectively.
4. Replacing x by 1 and y by x in (i), we get
$$(1+x)^n = {}^nC_0 x^0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_r x^r + \dots + {}^nC_{n-1} x^{n-1} + {}^nC_n x^n = \sum_{r=0}^{n} {}^nC_r x^r$$
This is expansion of $(1 + x)^n$ in ascending powers of x.
The expression $(1 + x)^n$ can also be expanded in descending powers of x.
$$(1 + x)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} + {}^nC_2 x^{n-2} + \dots + {}^nC_r\, x^{n-r} + \dots + {}^nC_{n-1} x + {}^nC_n = \sum_{r=0}^{n} {}^nC_r x^{n-r}$$
The coefficient of $(r + 1)^{\text{th}}$ term in the expansion of $(1 + x)^n$ is ${}^nC_r$.

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The coefficient of x^r in the expansion of $(1 + x)^n$ is ${}^nC_r$.

5. Replacing x by 1 and y by $-x$ in (i), we get

$$(1-x)^n = {}^nC_0x^0 - {}^nC_1x^1 + {}^nC_2x^2 - .... + {}^nC_r(-1)^r x^r$$
$$+ .... + {}^nC_{n-1}(-1)^{n-1}x^{n-1} + {}^nC_n(-1)^n x^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r x^r$$

PROPERTIES OF BINOMIAL EXPANSION $(x + y)^n$

(i) There are $(n + 1)$ terms in the expansion.

(ii) If the first term is x^n and the last term is y^n, then in the expansion, index of x decreases by one from left to right and index of y increases by one from left to right.

(iii) In any term, the suffix of C is equal to the index of y and the index of $x = n -$ (suffix of C), when the expansion is expanded in descending powers of x.

(iv) In each term, sum of the indices of x and y is equal to n or we can say that expansion is a homogeneous equation of degree n.

(v) Since ${}^nC_r = {}^nC_{n-r}$, we have ${}^nC_0 = {}^nC_n = 1$, ${}^nC_1 = {}^nC_{n-1} = n$ and ${}^nC_2 = {}^nC_{n-2} = \frac{n(n-1)}{2!}$ and so on. It follows that the binomial coefficient of the term equidistant from the beginning and from the end in the expansion, are equal.

GENERAL TERM

The term ${}^nC_r\, x^{n-r} y^r$ is the $(r + 1)^{\text{th}}$ term from beginning in the expansion of $(x + y)^n$. It is usually called the general term and it is denoted by T_{r+1}.

i.e., $T_{r+1} = {}^nC_r\, x^{n-r} y^r$.

Expansion	Equivalent Σ notation	General Term
$(x + y)^n$	$\sum_{r=0}^{n} {}^nC_r\, x^{n-r}y^r$	$T_{r+1} = {}^nC_r\, x^{n-r} y^r$
$(x - y)^n$	$\sum_{r=0}^{n} (-1)^r\, {}^nC_r x^{n-r}y^r$	$T_{r+1} = {}^nC_r(-1)^r x^{n-r} y^r$
$(1 + x)^n$	$\sum_{r=0}^{n} {}^nC_r\, x^r$	$T_{r+1} = {}^nC_r\, x^r$
$(1 - x)^n$	$\sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^r$	$T_{r+1} = (-1)^r\, {}^nC_r\, x^r$

p^{th} TERM FROM THE END

p^{th} term from the end in the expansion of $(x + y)^n$ is same as $(n - p + 2)^{\text{th}}$ term from the beginning.

MIDDLE TERMS

The middle terms depends upon the value of n.

(a) When n is even :

Total number of terms in the expansion of $(x + y)^n$ is $n + 1$ (odd). So there is only one middle term *i.e.* $\left(\frac{n}{2}+1\right)^{\text{th}}$ term and its value is given by

$$T_{\left(\frac{n}{2}+1\right)} = {}^nC_{n/2}\, x^{n/2}\, y^{n/2}.$$

(b) When n is odd :

Total number of terms in the expansion of $(x + y)^n$ is $n + 1$ (even). So there are two middle terms *i.e.* , $\left(\frac{n+1}{2}\right)^{\text{th}}$ and $\left(\frac{n+3}{2}\right)^{\text{th}}$ and they are given by

$$T_{\frac{n+1}{2}} = T_{\left(\frac{n-1}{2}+1\right)} = {}^nC_{\frac{n-1}{2}} x^{\frac{n+1}{2}} . y^{\frac{n-1}{2}}$$

and $$T_{\frac{n+3}{2}} = T_{\left(\frac{n+1}{2}+1\right)} = {}^nC_{\frac{n+1}{2}} x^{\frac{n-1}{2}} . y^{\frac{n+1}{2}}$$

GREATEST TERM

Greatest term means that the term which has greatest numerical value (irrespective of the sign).

If T_r and T_{r+1} be the r^{th} and $(r + 1)^{\text{th}}$ term in the expansion of $(1 + x)^n$, then $\frac{T_{r+1}}{T_r} = \frac{{}^nC_r x^r}{{}^nC_{r-1}x^{r-1}} = \frac{n-r+1}{r}x$

Let numerically, T_{r+1} be the greatest term in the above expansion, then $T_{r+1} \geq T_r$ or $\frac{T_{r+1}}{T_r} \geq 1$

$$\therefore \quad \frac{n-r+1}{r}|x| \geq 1 \Rightarrow r \leq \frac{(n+1)}{(1+|x|)}|x| \qquad \text{... (i)}$$

Now substituting values of n and x in (i), we get

$r \leq m + f$ or $r \leq m$

where m is a positive integer and f is a fraction such that $0 < f < 1$.

In the first case T_{m+1} is the greatest term while in the second case T_m and T_{m+1} are the greatest terms and both are equal.

Short Cut Method : To find the greatest term (numerically) in the expansion of $(1 + x)^n$.

(a) Calculate $m = \dfrac{|x|(n+1)}{|x|+1}$

(b) If m is not an integer, then $T_{[m]+1}$ is the greatest term, where [.] denotes the greatest integer less than or equal to m.

(c) If m is an integer then T_m & T_{m+1} are greatest terms and their values are equal.

HOW TO FIND GREATEST TERM IN THE EXPANSION OF $(x + y)^n$

Since $(x + y)^n = x^n(1 + y/x)^n$. Then find the greatest term in the expansion $\left(1+\dfrac{y}{x}\right)^n$.

GREATEST BINOMIAL COEFFICIENT IN THE EXPANSION OF $(x + y)^n$

(a) If n is even, then greatest binomial coefficient is the binomial coefficient of middle term *i.e.*, ${}^nC_{n/2}$

(b) If n is odd, then greatest binomial coefficients are the coefficient of the middle terms *i.e.*,

${}^nC_{\frac{(n-1)}{2}}$ and ${}^nC_{\frac{(n+1)}{2}}$

PROPERTIES OF BINOMIAL COEFFICIENTS

1. (i) ${}^nC_r = C_r = C(n, r) = \dbinom{n}{r} = {}^nC_{n-r} = \dfrac{n!}{r!(n-r)!}$

(ii) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(iii) ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$ and

${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$

(iv) ${}^nC_r = \dfrac{n}{r}.{}^{n-1}C_{r-1} = \dfrac{n}{r}.\dfrac{n-1}{r-1}.{}^{n-2}C_{r-2}$

(v) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$

(vi) ${}^nC_r$ is greatest when $\begin{cases} r = \dfrac{n}{2}, & \text{if } n \text{ is even} \\ r = \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \end{cases}$

(vii) $C_0 + C_1 + C_2 + \ldots.. + C_n = \sum_{r=0}^{n} C_r = 2^n$

(viii) $C_0 + C_2 + C_4 + \ldots.. = C_1 + C_3 + C_5 + \ldots.. = 2^{n-1}$

(ix) $C_0 - C_1 + C_2 - C_3 + C_4 - \ldots.. + (-1)^n C_n$

$= \sum_{r=0}^{n} (-1)^n C_r = 0$

2. (i) $\sum_{r=0}^{n-1} {}^{n-1}C_r = 2^{n-1}$ (ii) $\sum_{r=2}^{n} {}^{n-2}C_{r-2} = 2^{n-2}$

(iii) $\sum_{r=0}^{n-2} {}^{n-2}C_r = 2^{n-2}$ (iv) $\sum_{r=1}^{n} {}^{n-1}C_{r-1} = 2^{n-1}$

(v) $\sum_{r=1}^{n} {}^nC_r = 2^n - 1$ (vi) $\sum_{r=2}^{n} {}^nC_r = 2^n - n - 1$

(vii) $\sum_{r=1}^{n} (-1)^r\, {}^{n-1}C_{r-1} = 0$

(viii) $\sum_{r=2}^{n} (-1)^r\, {}^{n-2}C_{r-2} = 0$

3. (i) $r^2 = r(r-1) + r$

(ii) $r^3 = r(r-1)(r-2) + 3r(r-1) + r$

(iii) $r^4 = r(r-1)(r-2)(r-3) + 6r(r-1)(r-2) + 7r(r-1) + r$

4. (i) $\sum_{r=0}^{n} r \cdot {}^nC_r = n \cdot 2^{n-1}$

(ii) $\sum_{r=0}^{n} r(r-1)\, {}^nC_r = n(n-1)\, 2^{n-2}$

(iii) $\sum_{r=0}^{n} r(r-1)(r-2)\, {}^nC_r = n(n-1)(n-2)\, 2^{n-3}$

(iv) $\sum_{r=0}^{n} r^2\, {}^nC_r = n(n-1)\, 2^{n-2} + n.2^{n-1}$

(v) $\sum_{r=0}^{n} r^3\, {}^nC_r = n(n-1)(n-2)\, 2^{n-3} + 3n(n-1)2^{n-2} + n \cdot 2^{n-1}$

5. (i) $\sum_{r=0}^{n} \dfrac{{}^nC_r}{r+1} = \dfrac{2^{n+1}-1}{n+1}$

(ii) $\sum_{r=0}^{n} (-1)^r \dfrac{{}^nC_r}{r+1} = \dfrac{1}{n+1}$

(iii) $\sum_{r=0}^{n} \dfrac{{}^nC_r}{r+2} = \dfrac{1+n \cdot 2^{n+1}}{(n+1)(n+2)}$

6. (i) $\sum_{r=0}^{n} ({}^nC_r)^2 = {}^{2n}C_n$

(ii) $\sum_{r=0}^{n} (-1)^r ({}^nC_r)^2 = \begin{cases} 0\ , & \text{if } n \text{ is odd} \\ (-1)^{n/2}.\,{}^nC_{n/2}, & \text{if } n \text{ is even} \end{cases}$

7. (i) $\sum_{r=1}^{n} r = \dfrac{n(n+1)}{2}$

(ii) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $\sum_{r=1}^{n} r^3 = \left(\frac{n(n+1)}{2}\right)^2$

(iv) $\sum_{r=0}^{n-1} a^r = \frac{1(a^n-1)}{a-1}$

PROBLEMS

Single Correct Answer Type

1. The expression $\{x + (x^3 - 1)^{1/2}\}^5 + \{x - (x^3 - 1)^{1/2}\}^5$ is a polynomial of degree

(a) 5 (b) 6 (c) 7 (d) 8

2. The coefficient of x^{3l+2} in the expansion of $(a + x)^l (b + x)^{l+1} (c + x)^{l+2}$ must be, (l is a positive integer)

(a) $l(a + b + c)$

(b) $l(a + b + c) + b + 2c$

(c) $l(a + b + c) + a + 2b + 3c$

(d) None of these

3. The number of integral terms in the expansion of $(\sqrt{3}+\sqrt[8]{5})^{256}$ is

(a) 35 (b) 32 (c) 33 (d) 34

4. The number of terms in the expansion $(1 + x)^{101}(1 + x^2 - x)^{100}$, is

(a) 302 (b) 301 (c) 202 (d) 101

5. The value of

$^nC_0 \cdot {}^{2n}C_r - {}^nC_1 \cdot {}^{2n-2}C_r + {}^nC_2 \cdot {}^{2n-4}C_r + \ldots + (-1)^n\, {}^nC_n \cdot {}^{2n-2n}C_n$, is equal to

(a) $\begin{cases} 2^{2n-r}\cdot {}^nC_{r-n}, & \text{if } r > n \\ 0, & \text{if } r < n \end{cases}$

(b) $\begin{cases} 3^{2n-r}\cdot {}^nC_{r-n}, & \text{if } r > n \\ 0, & \text{if } r < n \end{cases}$

(c) $\begin{cases} 2^{2n+r}\cdot {}^nC_{r+n}, & \text{if } r > n \\ 0, & \text{if } r < n \end{cases}$

(d) None of these

6. The value of $\sum_{r=0}^{s}\sum_{\substack{s=1 \\ r\le s}}^{n} {}^nC_s\, {}^sC_r$, is

(a) $3^n - 1$ (b) $3^n + 1$

(c) 3^n (d) None of these

7. If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n}$, then

$\begin{vmatrix} a_{n-3} & a_{n-1} & a_{n+1} \\ a_{n-6} & a_{n-3} & a_{n+3} \\ a_{n-14} & a_{n-7} & a_{n+7} \end{vmatrix}$ is

(a) 1 (b) 2

(c) 0 (d) None of these

8. If $b_1, b_2, \ldots, b_n$ are the n^{th} roots of unity, then $^nC_1 \cdot b_1 + {}^nC_2 \cdot b_2 + \ldots + {}^nC_n \cdot b_n$ is equal to

(a) $\frac{b_1}{b_2}$ (b) $\frac{b_1}{b_2}\{(b_1 + b_2)^{2n} - 1\}$

(c) $\frac{b_1}{b_2}\{(1 + b_2)^n - 1\}$ (d) None of these

9. The value of $C_3 + C_7 + C_{11} + \ldots$ is

(a) $\frac{1}{2}\left(2^{n-1} - 2^{n/2}\sin\frac{n\pi}{4}\right)$

(b) $\frac{1}{2}\left(2^{n-1} + 2^{n/2}\sin\frac{n\pi}{4}\right)$

(c) $\frac{1}{4}\left(2^{n+1} - 2^{n/2}\sin\frac{n\pi}{4}\right)$

(d) None of these

10. The integer just greater than $(\sqrt{3}+1)^{2m}$ contains

(a) 2^{m+2} as a factor

(b) 2^{m+1} as a factor

(c) 2^{m+3} as a factor

(d) None of these

11. In a triangle ABC, the value of the expression

$\sum_{r=0}^{n} {}^nC_r a^r b^{n-r} \cos(rB - (n-r)A)$ is

(a) $(a + b)^n$ (b) c^n

(c) $(\sin A + \sin B)^n$ (d) $(\sin C)^n$

12. The value of the expression

$\sum_{k=1}^{n-1} {}^nC_k[\cos kx \cdot \cos(n+k)x + \sin(n-k)x \cdot \sin(2n-k)x]$

is

(a) $(2^n - 2)\cos nx$ (b) $(2^n - 2)\sin nx$

(c) $(2^n - 1)\cos nx$ (d) $(2^n - 1)\sin nx$

13. If $^{n-1}C_r = (k^2 - 3)\, {}^nC_{r+1}$, then $k \in$

(a) $(-\infty, -2]$ (b) $[2, \infty)$

(c) $[-\sqrt{3}, \sqrt{3}]$ (d) $(\sqrt{3}, 2]$

Multiple Correct Answer Type

14. The coefficient of three consecutive terms in the expansion of $(1 + x)^n$ are in the ratio 1 : 7 : 42, then
(a) n is divisible by 5
(b) n is divisible by 7
(c) n is divisible by 11
(d) $n = 45$

15. If the expansion of $\left(x-\frac{a}{x}\right)^n$ and $\left(x+\frac{b}{x^2}\right)^n$ in powers of x has one term independent of x, then n must be divisible by
(a) 6 (b) 4 (c) 3 (d) 2

16. The value of ${}^{2n}C_0^2 - {}^{2n}C_1^2 + {}^{2n}C_2^2 - .... + {}^{2n}C_{2n}^2$ must be
(a) even (b) odd
(c) ${}^{2n}C_n$ (d) $(-1)^n\ {}^{2n}C_n$

17. Let $A = (1 + x)(1 + x + x^2)(1 + x + x^2 + x^3)\ ....\ (1 + x + x^2 + ... + x^n)$
(a) sum of the coefficients in $A = n!$
(b) sum of the coefficients in $A = (n + 1)!$
(c) the highest power of x is $\frac{n(n+1)}{2}$
(d) the coefficient of x is n

18. The value of the sum $\sum_{p=1}^{n}\sum_{m=p}^{n} {}^nC_m\, {}^mC_p$ must be
(a) $3^n + 2^n$ (b) $3^n - 2^n$
(c) 6^n (d) Independent of p

19. In the expansion of $\left(\sqrt[3]{4}+\frac{1}{\sqrt[4]{6}}\right)^{20}$
(a) Number of irrational terms are 17
(b) Number of irrational terms are 19
(c) Number of rational terms are 2
(d) Middle term is irrational

20. The greatest value of the term independent of x in the expansion of $\left(x\sin\alpha+\frac{\cos\alpha}{x}\right)^{10}$, where $\alpha \in R$, is
(a) $\frac{10!}{(5!)^2}$ (b) $\frac{10!}{(5!)^2}\cdot\frac{1}{2^5}$
(c) rational (d) less than 8

21. 2^{60} leaves 1 as remainder when divided by
(a) 3 (b) 7 (c) 15 (d) 20

22. In the expansion of $\left[\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right]^{10}$
(a) independent term does not exist
(b) 5^{th} term is independent term
(c) coeff. of x^6 does not exist
(d) 2^{nd} term contains x^6

23. The value of ${}^nC_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + ... + {}^{n+k}C_k$ is equal to
(a) ${}^{n+k+1}C_k$ (b) ${}^{n+k+1}C_{n+1}$
(c) ${}^{n+k}C_{n+1}$ (d) None of these

24. The values of r satisfying the equation ${}^{69}C_{3r-1} - {}^{69}C_{r^2} = {}^{69}C_{r^2-1} - {}^{69}C_{3r}$, is
(a) 1 (b) 2 (c) 3 (d) 7

25. If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$, then the value of $a_0 + a_3 + a_6 + ...$, is
(a) $a_1 + a_4 + a_7 + ...$
(b) $a_2 + a_5 + a_8 + ...$
(c) 3^{n-1}
(d) None of these

26. If the term independent of x in the expansion of $\left(\sqrt{x}-\frac{k}{x^2}\right)^{10}$ is 405, then the value of k must be
(a) 3 (b) -3 (c) 9 (d) -9

27. In the expansion of $\left(x+\frac{a}{x^2}\right)^n$ $(a \neq 0)$ if term independent of x does not exist, then n must be
(a) 20 (b) 16 (c) 15 (d) 10

Comprehension Type

Paragraph for Q. No. 28 and 29

If A and B are positive integers and t is a positive integer which is not a perfect square, then the number $(A+B\sqrt{t})$ and its positive integral powers are essentially irrational. If n is a positive integer, then it can be noticed that $(A+B\sqrt{t})^n+(A-B\sqrt{t})^n$ is a positive integer E. Further if $0<(A-B\sqrt{t})^n<1$, then E is the integer just next to $(A+B\sqrt{t})^n$. From the equality $(A+B\sqrt{t})^n+(A-B\sqrt{t})^n=E$ it can also be concluded that sum of fractional parts is equal to 1. We can similarly draw conclusions when $(A\sqrt{t}+B)^n-(A\sqrt{t}-B)^n$ is a positive integer, where $0 < (A\sqrt{t}-B)^n < 1$.

28. If $[x]$ denotes the greatest integer $\leq x$ and $g=(3+\sqrt{5})^n+(3-\sqrt{5})^n$ then $[g]$ must be equal to

(a) $(3+\sqrt{5})^n+(3-\sqrt{5})^n+1$

(b) $(3+\sqrt{5})^n+(3-\sqrt{5})^n-1$

(c) $(3+\sqrt{5})^n+(3-\sqrt{5})^n$

(d) $(3+\sqrt{5})^n-(3-\sqrt{5})^n$

29. Let $u_n=(\sqrt{3}+1)^{2n}+(\sqrt{3}-1)^{2n}$ then

(a) $4u_{n+1} = 2u_n - 3u_{n-1}$

(b) $u_{n+1} = 4u_n - u_{n-1}$

(c) $u_{n+1} = 8u_n - 4u_{n-1}$

(d) $u_{n+1} = 2u_n$

Paragraph for Q. No. 30 and 31

If $(1+x)^n=\sum_{r=0}^{n}{}^nC_r x^r$, then the sum of the binomial coefficients can be obtained by substituting $x = 1$. But in some case we have to find the sum of coefficients in some particular order, we can substitute x by $\pm ix$ or $\pm \omega x$ or $\pm \omega^2 x$ depending upon the requirements.

30. The sum of binomial coefficients $C_0 + C_4 + C_8 + ...$ must be equal to

(a) $2^{n/2}\cos\left(\frac{\pi}{8}\right)$ (b) $2^{n/2}\sin\left(\frac{n\pi}{8}\right)$

(c) $2^n+2^{n/2}\cos\left(\frac{n\pi}{4}\right)$

(d) $2^{n-2}+2^{\frac{n}{2}-1}\cos\left(\frac{n\pi}{4}\right)$

31. The sum of the binomial coefficients ${}^nC_0 + {}^nC_3 + {}^nC_6 + ...$ must be equal to

(a) $\frac{2^n}{3}$

(b) $\frac{1}{3}\left(2^n+\cos\left(\frac{n\pi}{3}\right)\right)$

(c) $\frac{1}{3}\left(2^n+2\cos\frac{n\pi}{3}\right)$

(d) $\frac{1}{3}\left(2^n+2\sin\left(\frac{n\pi}{3}\right)\right)$

Integer Answer Type

32. The number of solutions in non-negative integers of the equation $x_1 + x_2 + ... + x_6 = 8$ is same as number of integer solutions of $x_1 + x_2 + ... + x_9 = k$, where k is an integer, then k must be

33. If $x > 0$ then values of x for which fourth term of the expansion of $\left(2+\frac{3x}{8}\right)^{10}$ has greatest value lie in the interval $\left(2,\frac{64}{\lambda}\right)$, the numerical $\lambda/3$ must be

34. If $\sum_{k=1}^{n}k^3\left(\frac{{}^nC_k}{{}^nC_{k-1}}\right)^2=\frac{n(n+1)^2(n+2)}{\lambda}$ then the sum of digits of λ must be

35. For every even positive integer n, $20^n + 16^n - 3^n - 1$ is divisible by product of two distinct primes. Each of even is less than 20. The unit digit of product of two primes be

SOLUTIONS

1. (c) : $(x+\sqrt{x^3-1})^5+(x-\sqrt{x^3-1})^5$

$=2\left[x^5+{}^5C_2x^3\left(\sqrt{x^3-1}\right)^2+{}^5C_4x\left(\sqrt{x^3-1}\right)^4\right]$

which will clearly be a polynomial of degree 7 when simplified.

2. (b) : Given expression = $\{(x + a)(x + a) .... l \text{ times}\} \times \{(x + b)(x + b) ..... (l + 1) \text{ times}\} \times \{(x + c)(x + c) ..... (l + 2) \text{ times}\}$

$= x^{3l+3} + [al + b(l + 1) + c(l + 2)]x^{3l+2} + ...$

$\Rightarrow$ Coefficient of x^{3l+2} is $(a + b + c)l + b + 2c$

$\Rightarrow$ Choice (b) is correct.

3. (c) : $T_{r+1}={}^{256}C_r\left(\sqrt{3}\right)^{256-r}\left(\sqrt[8]{5}\right)^r={}^{256}C_r\,3^{\frac{256-r}{2}}\,5^{\frac{r}{8}}$

For a term to be integer it is necessary and sufficient that r is a multiple of 8.

$\Rightarrow r = 0, 8, 16, 24, ..., 256.$

which is an AP containing 33 terms.

4. (c) : Here $(1 + x)^{101} (1 + x^2 - x)^{100}$

$= (1 + x)(1 + x^3)^{100}$

$= (1 + x)\{C_0 + C_1x^3 + C_2x^6 + ... + C_{100}\,x^{300}\}$

$= C_0 + C_0x + C_1x^3 + C_1x^4 + ... + C_{100}x^{300} + C_{100}\,x^{301}$

$\therefore$ Total no. of terms = 101 + 101 = 202

5. (a) : We have,

${}^nC_0 \cdot {}^{2n}C_r - {}^nC_1 \cdot {}^{2n-2}C_r + {}^nC_2 \cdot {}^{2n-4}C_r + ...$
$...... + (-1)^n\,{}^nC_n \cdot {}^{2n-2n}C_n$

= Coefficient of x^r in

$[{}^nC_0(1 + x)^{2n} - {}^nC_1(1 + x)^{2n-2} + {}^nC_2(1 + x)^{2n-4} - .... + (-1)^n\,{}^nC_n(1 + x)^{2n-2n}]$

= Coefficient of x^r in $[(1 + x)^2 - 1]^n$
= Coefficient of x^r in $(2x + x^2)^n$
= Coefficient of x^{r-n} in $(2 + x)^n$

$$= \begin{cases} 2^{2n-r} \cdot {}^nC_{r-n}; & \text{if } r > n \\ 0 \quad ; & \text{if } r < n \end{cases}$$

6. (a) : We have, $\sum_{r=0}^{s}\sum_{s=1}^{n} {}^nC_s {}^sC_r$

$$= \sum_{s=1}^{n} {}^nC_s({}^sC_0 + {}^sC_1 + {}^sC_2 + \ldots.. + {}^sC_s) \ [\because r \le s]$$

$$= \sum_{s=1}^{n} {}^nC_s(2)^s = \sum_{s=1}^{n} {}^nC_s 2^n - {}^nC_0 2^0$$

$= (1 + 2)^n - 1 = 3^n - 1$

7. (c) : Given

$(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots.. + a_{2n}x^{2n}$

Replacing x by $1/x$, we get

$(1 + x + x^2)^n = a_0x^{2n} + a_1x^{2n-1} + a_2 x^{2n-2} + \ldots. + a_{2n}$

$\Rightarrow a_0 + a_1x + a_2x^2 + \ldots. + a_{2n} x^{2n}$
$= a_0 x^{2n} + a_1x^{2n-1} + \ldots. + a_{2n}$

On equating the coefficients of similar powers of x, we get

$a_0 = a_{2n}, a_1 = a_{2n-1}, \ldots.., a_r = a_{2n-r}$

$\therefore$ The value of given determinant is equal to 0 as 2nd and 3rd columns are identical.

8. (c) : As $b_1, b_2, \ldots.., b_n$ are n^{th} roots of unity

$\Rightarrow b_1, b_2, \ldots., b_n$ are in GP

where $b_1 = 1$, $b_2 = e^{i2\pi/n}$,

$b_3 = e^{i4\pi/n}, \ldots..., b_n = e^{i\frac{2(n-1)\pi}{n}}$

Clearly b_2 is common ratio and $b_n = b_1(b_2)^{n-1}$.

$\therefore$ Given expression

$= {}^nC_1 \cdot b_1 + {}^nC_2 \cdot b_2 + \ldots. + {}^nC_n \cdot b_n$
$= b_1\{{}^nC_1 + {}^nC_2b_2 + {}^nC_3(b_2)^2 + \ldots.. + {}^nC_n(b_2)^{n-1}\}$

$$= \frac{b_1}{b_2}\{{}^nC_1b_2 + {}^nC_2(b_2)^2 + \ldots. + {}^nC_n(b_2)^n\}$$

$$= \frac{b_1}{b_2}\{(1+b_2)^n - 1\}$$

9. (a) : We have,

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots.. + C_nx^n$ ... (1)

On putting $x = 1$ and -1, respectively in (1), we get

$C_0 + C_1 + C_2 + \ldots.. + C_n = 2^n$
$C_0 - C_1 + C_2 - C_3 + \ldots.. + (-1)^n C_n = 0$

On adding and subtracting these two, we get

$C_0 + C_2 + C_4 + \ldots... = 2^{n-1}$ ... (2)

and $C_1 + C_3 + C_5 + \ldots.. = 2^{n-1}$ ... (3)

On putting $x = i$ in (1), we get

$(C_0 - C_2 + C_4 - C_6 + \ldots..)$
$+ i(C_1 - C_3 + C_5 - \ldots..) = (1 + i)^n$

$\Rightarrow (C_0 - C_2 + C_4 - C_6 + \ldots..)$
$+ i(C_1 - C_3 + C_5 - C_7 + \ldots..)$

$$= 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$$

On equating imaginary parts on both sides, we get

$$C_1 - C_3 + C_5 - C_7 + \ldots. = 2^{n/2} \sin\frac{n\pi}{4} \quad \ldots (4)$$

On subtracting Eqs. (4) from (3), we get

$$2(C_3 + C_7 + C_{11} + \ldots.) = \left(2^{n-1} - 2^{n/2}\sin\frac{n\pi}{4}\right)$$

$$\Rightarrow C_3 + C_7 + C_{11} + \ldots.. = \frac{1}{2}\left(2^{n-1} - 2^{n/2}\sin\frac{n\pi}{4}\right)$$

10. (b) : Let $(\sqrt{3}+1)^{2m} = I + f$, where $I \in N$ and $0 < f < 1$.

Let $G = (\sqrt{3}-1)^{2m}$. Then,

$I + f + G = (\sqrt{3}+1)^{2m} + (\sqrt{3}-1)^{2m}$
$= 2^m(2+\sqrt{3})^m + 2^m(2-\sqrt{3})^m$
$= 2^{m+1} \times$ an integer ...(1)

$\Rightarrow I + f + G$ = an even integer
$\Rightarrow f + G$ = an even integer $- I$
$\Rightarrow f + G$ = an integer
$\Rightarrow f + G = 1$ $[\because 0 < f < 1, 0 < G < 1]$

On putting $f + G = 1$ in (1), we get

$I + 1 = 2^{m+1} \times$ an integer

$\Rightarrow 2^{m+1}$ is a factor of the integer just greater than $(\sqrt{3}+1)^{2m}$.

11. (b) : $\sum_{r=0}^{n} {}^nC_r a^r b^{n-r} \cos(rB - (n-r)A)$

$$= \text{real part of } \sum_{r=0}^{n} {}^nC_r a^r b^{n-r} e^{i\{rB-(n-r)A\}}$$

Now, $\sum_{r=0}^{n} {}^nC_r a^r b^{n-r} e^{i\{rB-(n-r)A\}}$

$$= \sum_{r=0}^{n} {}^nC_r (ae^{iB})^r (be^{-iA})^{n-r} = (ae^{iB} + be^{-iA})^n$$

$= (a \cos B + ia \sin B + b \cos A - bi \sin A)^n$
$= \{(a \cos B + b \cos A) + i(a \sin B - b \sin A)\}^n$
$= \{c + i \cdot 0\}^n = c^n$

12. (a) : Let the given expression be E, then E can be written as,

$$E=\sum_{k=1}^{n-1}{}^nC_k\cos kx\cdot\cos(n+k)x$$
$$+\sum_{k=1}^{n-1}{}^nC_k\sin(n-k)x\cdot\sin(2n-k)x$$

or $$E=\sum_{k=1}^{n-1}{}^nC_k\cos kx\cos(n+k)x$$
$$+\sum_{k=1}^{n-1}{}^nC_k\sin(k)x\cdot\sin(n+k)x$$

{replacing k by $(n-k)$ in the second sum and using ${}^nC_k={}^nC_{n-k}$}

$$E=\sum_{k=1}^{n-1}{}^nC_k(\cos kx\cos(n+k)x$$
$$+\sin kx\cdot\sin(n+k)x)$$
$$=\sum_{k=1}^{n-1}{}^nC_k\cos nx$$
$= \cos nx\{({}^nC_0+{}^nC_1+.....+{}^nC_n)-{}^nC_0-{}^nC_n\}$

$= \cos nx\{2^n-2\}$

$\therefore\ E=(2^n-2)\cos nx$

13. (d) : Here, ${}^{n-1}C_r=(k^2-3)\,{}^nC_{r+1}$

$$\Rightarrow\ {}^{n-1}C_r=(k^2-3)\frac{n}{r+1}\,{}^{n-1}C_r$$

$$\Rightarrow\ k^2-3=\frac{r+1}{n}$$

(Since, $n-1\ge r\ \Rightarrow\ \frac{r+1}{n}\le 1$ and $n, r>0$)

$\Rightarrow\ 0<k^2-3\le 1$ or $3<k^2\le 4$

$\Rightarrow\ k\in\left[-2,-\sqrt{3}\right)\cup\left(\sqrt{3},2\right]$

14. (a, c) : The condition leads to

${}^nC_r:{}^nC_{r+1}:{}^nC_{r+2}=1:7:42$

On solving by usual techniques of cancelling factorials we will get $n=55$

Whence choices (a) and (c) become true.

15. (a, c, d) : In the first expansion

$$T_{r+1}={}^nC_rx^{n-r}\left(-\frac{a}{x}\right)^r={}^nC_r(-a)^rx^{n-2r}$$

In the second expansion

$$T_{r+1}={}^nC_rx^{n-r}\left(\frac{b}{x^2}\right)^r={}^nC_rb^rx^{n-3r}$$

We must have $n-2r=0,\ n-3r=0$

$$\Rightarrow\ r=\frac{n}{2},\ r=\frac{n}{3}$$

$\Rightarrow\ n$ must be divisible by 2 and 3 both $\Rightarrow n$ is divisible by 6

$\Rightarrow$ Choice (a), (c) and (d) are correct.

16. (a,d) : If we multiply the expansion of $(1+x)^{2n}$ and $(x-1)^{2n}$ and compare the coefficient of x^{2n}, we get

$${}^{2n}C_0^2-{}^{2n}C_1^2+{}^{2n}C_2^2-.....+{}^{2n}C_{2n}^2=(-1)^n\,{}^{2n}C_n$$

Since ${}^{2n}C_n$ is always even. Choices (a), (d) are correct.

17. (b, c, d) : Sum of the coefficient is obtained by putting $x=1$ which is (2) (3) (4) .... $(n+1)=(n+1)!$

$\Rightarrow$ Choice (b) is true and choice (a) is false.

The highest power of x will obtained when all last terms get multiplied.

i.e., $1+2+3+....+n=\frac{n(n+1)}{2}$

$\Rightarrow$ Choice (c) is true

Coefficient of $x=1+1+1+......\ n$ times $=n$

18.(b, d) : (d) is obviously correct since p is the index of the summation.

Now summation $=\sum_{p=1}^{n}\sum_{m=p}^{n}{}^nC_m\,{}^mC_p$

$$=\sum_{p=1}^{n}({}^nC_p\,{}^pC_p+{}^nC_{p+1}\,{}^{p+1}C_p$$
$$+{}^nC_{p+2}\,{}^{p+2}C_p+.......+{}^nC_n{}^nC_p)$$

Now, ${}^nC_p\,{}^pC_p+{}^nC_{p+1}\,{}^{p+1}C_p$
$+\,{}^nC_{p+2}\,{}^{p+2}C_p+.....+{}^nC_n\,{}^nC_p$

= Coeff. of x^p is $[{}^nC_p(1+x)^p+{}^nC_{p+1}(1+x)^{p+1}$
$+\,{}^nC_{p+2}(1+x)^{p+2}+.....+{}^nC_n(1+x)^n]$

= Coeff. of x^p is $[{}^nC_0+{}^nC_1(1+x)^1+{}^nC_2(1+x)^2+....$
${}^nC_p(1+x)^p+{}^nC_{p+1}(1+x)^{p+1}+.....+{}^nC_n(1+x)^n]$

(Since first p terms have no term of x^p therefore adding them does not affect the coefficient)

= Coeff. of x^p is $[1+(1+x)]^n$

= Coeff. of x^p is $(2+x)^n={}^nC_p2^{n-p}$

$\Rightarrow$ Summation

$$=\sum_{p=1}^{n}{}^nC_p2^{n-p}=2^n\sum_{p=1}^{n}{}^nC_p\left(\frac{1}{2}\right)^p$$

$$=2^n\left[\left(1+\frac{1}{2}\right)^n-1\right]=3^n-2^n$$

19. (b, c, d) : $T_{r+1}={}^{20}C_r\,2^{\frac{160-11r}{12}}\cdot 3^{\left(-\frac{r}{4}\right)}$

For rational terms, $\frac{r}{4}$ and $\frac{160-11r}{12}$ must be integer.

$\frac{r}{4}$ is integer for $r=0,4,8,12,16,20$ but $\frac{160-11r}{12}$ is

integer for $r = 8, 20$.

$\Rightarrow$ Number of rational terms = 2

Remaining $(21 - 2) = 19$ terms are irrational.

20. (b, c, d) : $T_{r+1} = {}^{10}C_r\,(x\sin\alpha)^{10-r}\left(\dfrac{\cos\alpha}{x}\right)^r$

For independent term $10 - 2r = 0 \Rightarrow r = 5$

$\therefore\quad T_{r+1} = {}^{10}C_5\dfrac{1}{2^5}(\sin 2\alpha)^5 < {}^{10}C_5\cdot\dfrac{1}{2^5}$

$\Rightarrow$ Greatest value is ${}^{10}C_5\cdot\dfrac{1}{2^5}$ which is less than 8.

21. (a, b, c) : $2^{60} = (2^2)^{30} = (1 + 3)^{30}$

$= 1 + {}^{30}C_1\,3^1 + {}^{30}C_2\,3^2 + ...$

It will give one as remainder when divided by 3, 7, 15 as

$2^{60} = (2^3)^{20} = (1 + 7)^{20} = 1 + {}^{20}C_1\,7^1 + {}^{20}C_2 7^2 + ...$

$2^{60} = (2^4)^{15} = (1 + 15)^{15}$

$= 1 + {}^{15}C_1(15)^1 + {}^{15}C_2\,15^2 + ...$

22.(b, c) : The given expansion can be written as,

$$\left[\frac{(x^{1/3})^3+1^3}{x^{2/3}-x^{1/3}+1}-\frac{(x^{1/2})^2-1^2}{x^{1/2}(x^{1/2}-1)}\right]^{10} = \left[x^{1/3}-x^{-1/2}\right]^{10}$$

$T_{r+1} = {}^{10}C_r(-1)^r(x^{1/3})^{10-r}(x^{-1/2})^r$

For independent term $\dfrac{10-r}{3}-\dfrac{r}{2}=0$

$\Rightarrow\ r = 4$

Hence 5th term is independent of x.

By putting $\dfrac{10-r}{3}-\dfrac{r}{2}=6$

$\Rightarrow\ r \notin N$

23.(a, b) : The given expression is the coefficient of x^n in

$(1 + x)^n + (1 + x)^{n+1} + ... + (1 + x)^{n+k}$

$\Rightarrow$ Coefficient of x^n in $\left[(1+x)^n\left\{\dfrac{(1+x)^{k+1}-1}{1+x-1}\right\}\right]$

$\Rightarrow$ Coefficient of x^n in $\left[\dfrac{(1+x)^n(1+x)^{k+1}-(1+x)^n}{x}\right]$

$\Rightarrow$ Coefficient of x^{n+1} in $[(1 + x)^{n+k+1} - (1 + x)^n]$

i.e., ${}^{n+k+1}C_{n+1}$ or ${}^{n+k+1}C_k$

24. (c, d) : We are given that

${}^{69}C_{3r-1} + {}^{69}C_{3r} = {}^{69}C_{r^2-1} + {}^{69}C_{r^2}$

$\Rightarrow\ {}^{70}C_{3r} = {}^{70}C_{r^2}$

$\Rightarrow$ either $3r = r^2$ or $r^2 + 3r = 70$

i.e., either $r = 0, 3$ or $r = 7, -10$

But the given equation is not defined for $r = 0, -10$

Hence, $r = 3$ or 7.

25. (a, b, c) : Since $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$

Replacing x by ω, we have

$(1 + \omega + \omega^2)^n = a_0 + a_1\omega + a_2\omega^2 + ... + a_{2n}\omega^{2n}$ ....(1)

Again, replacing x by ω^2, we have

$(1 + \omega + \omega^2)^n = a_0 + a_1\omega^2 + a_2\omega + ... + a_{2n}\omega^{4n}$ ....(2)

On putting, $\omega = 1$ in (1), we have

$3^n = a_0 + a_1 + a_2 + ... + a_{2n}$ ....(3)

On adding (1), (2) and (3), we have

$3^n = 3(a_0 + a_3 + a_6 + ...)$

$\Rightarrow\ a_0 + a_3 + a_6 + ... = 3^{n-1}$ ....(4)

Now, subtracting (2) from (1), we have

$(\omega - \omega^2)(a_1 - a_2 + a_4 - a_5 + ...) = 0$ ....(5)

Again, subtracting (4) from (3), we get

$a_1 + a_2 + a_4 + a_5 + ...$

$= 3^n - 3^{n-1} = 2(3^{n-1})$ ....(6)

On adding (5) and (6), we get

$2(a_1 + a_4 + a_7 ....) = 2(3^{n-1})$

Again, subtracting (5) from (6), we get

$2(a_2 + a_5 + ...) = 2(3^{n-1})$

26. (a, b) : $T_{r+1} = {}^{10}C_r(-k)^r\,x^{5-\frac{5r}{2}}$

For term independent of x, $5-\dfrac{5r}{2} = 0\ \Rightarrow r = 2$

$\therefore\ {}^{10}C_2(-k)^2 = 405\ \Rightarrow\ k^2 = 9\ \Rightarrow k = \pm\,3.$

27. (a, b, d) : $T_{r+1} = {}^nC_r\,x^{n-r}\left(\dfrac{a}{x^2}\right)^r = {}^nC_r\,a^r x^{n-3r}$

If term independent of x does not exist, then $n - 3r$ must not be zero for positive integers n and r.

$\Rightarrow\ r = \dfrac{n}{3}$ must not be an integer.

Now, the integers given in choices (a), (b) and (d) are not divisible by 3.

So, (a), (b) and (d) are the correct choices.

Since 15 is divisible by 3, choice (c) is not true.

28. (c) : If we put $n = 1$, then

$g = (3+\sqrt{5})+(3-\sqrt{5}) = 6$

$\Rightarrow\ [g] = 6$

Now observe the table

Choice	Value of $n = 1$
(a)	7
(b)	5
(c)	6
(d)	$2\sqrt{5}$ (absurd)

Hence, it is concluded that choice (c) is correct.

29. (c) : $u_n = \left((\sqrt{3}+1)^2\right)^n + \left((\sqrt{3}-1)^2\right)^n$

$= (4+2\sqrt{3})^n + (4-2\sqrt{3})^n$

$= \alpha^n + \beta^n$ where $\alpha + \beta = 8$, $\alpha\beta = 4$

Now, $8u_n = 8(\alpha^n + \beta^n) = (\alpha + \beta)(\alpha^n + \beta^n)$

$= \alpha^{n+1} + \beta^{n+1} + \alpha\beta^n + \beta\alpha^n$

$= u_{n+1} + \alpha\beta(\alpha^{n-1} + \beta^{n-1})$

$= u_{n+1} + 4u_{n-1}$

Thus $u_{n+1} = 8u_n - 4u_{n-1}$

30. (d) : Since $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$

Replace x by ix, we have

$(1 + ix)^n = C_0 + iC_1x - C_2x^2 - iC_3x^3 + C_4x^4 + iC_5x^5 + \ldots.$

$(1 + ix)^n = (C_0 - C_2x^2 + C_4x^4 + \ldots.) + i(C_1x - C_3x^3 + C_5x^5 + \ldots.)$ ...(1)

Similarly, $(1 - ix)^n = (C_0 - C_2x^2 + C_4x^4 + \ldots.) - i(C_1x - C_3x^3 + C_5x^5 + \ldots.)$ ....(2)

On adding (1) and (2), we have

$(1 + ix)^n + (1 - ix)^n = 2(C_0 - C_2x^2 + C_4x^4 + \ldots.)$

Replace x by 1, we get

$2(C_0 - C_2 + C_4 - C_6 + C_8 - \ldots..)$

$= (1 + i)^n + (1 - i)^n = 2\cdot 2^{n/2}\cos\left(\frac{n\pi}{4}\right)$ ....(3)

Also, $C_0 + C_2 + C_4 + C_6 + C_8 + \ldots. = 2^{n-1}$ ....(4)

On adding (3) and (4), we have

$2(C_0 + C_4 + C_8 + C_{12} + \ldots.) = 2^{n-1} + 2^{n/2}\cos\left(\frac{n\pi}{4}\right)$

31. (c) : $C_0 + C_3 + C_6 + \ldots..$

$= \frac{1}{3}\{2^n + (-1)^n\omega^{2n} + (-1)^n\omega^n\}$

$= \frac{1}{3}\{2^n + (-1)^n(\omega^n + \omega^{2n})\}$

$= \frac{1}{3}\{2^n + (-1)^n(\bar{\omega}^n + \omega^n)\}$ $\{\because \omega^2 = \bar{\omega}\}$

$= \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right)$

32. [5] : We must have ${}^{6+8-1}C_{6-1} = {}^{k+9-1}C_{9-1}$

$\Rightarrow {}^{13}C_8 = {}^{k+8}C_8$ $[\because {}^{13}C_5 = {}^{13}C_8]$

which has an obvious solution $k = 5$.

33. [7] : Expression $= 2^{10}\left(1+\frac{3x}{16}\right)^{10}$

If T_4 is greatest then $T_4 > T_5$, $T_4 > T_3$

$\Rightarrow {}^{10}C_3\left(\frac{3x}{16}\right)^3 > {}^{10}C_4\left(\frac{3x}{16}\right)^4$

and ${}^{10}C_3\left(\frac{3x}{16}\right)^3 > {}^{10}C_2\left(\frac{3x}{16}\right)^2$

$\Rightarrow x < \frac{64}{21},\ x > 2$

$\Rightarrow x \in \left(2, \frac{64}{21}\right) \Rightarrow \lambda = 21$

Hence $\lambda/3 = 7$.

34. [3] : It can be easily verified that

$k^3\left(\frac{{}^nC_k}{{}^nC_{k-1}}\right)^2 = k(n + 1)^2 + k^3 - 2(n + 1)k^2$

$\sum k^3\left(\frac{{}^nC_k}{{}^nC_{k-1}}\right)^2 = (n + 1)^2\,\Sigma k - 2(n + 1)\,\Sigma k^2 + \Sigma k^3$

$= \frac{n(n+1)^2(n+2)}{12}$

$\Rightarrow \lambda = 12$

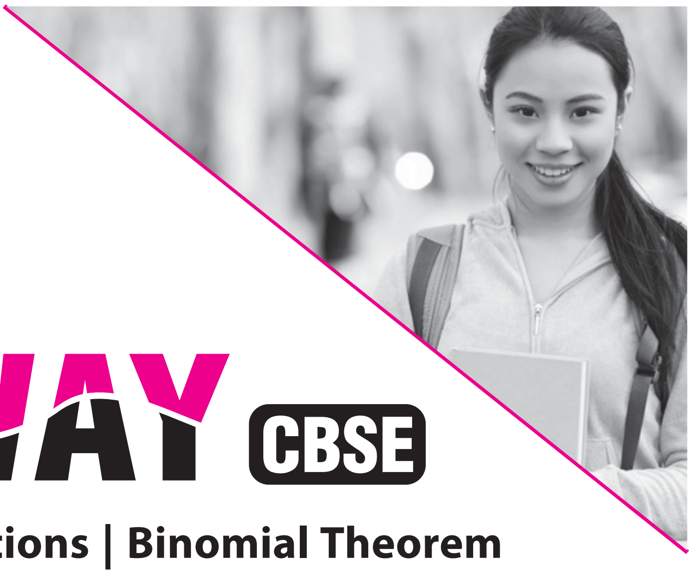
35. [3] : For $n = 2$, the given expression $= 646$

$= 2 \times 17 \times 19$

$\Rightarrow$ Odd primes are 17 and 19 and their product $= 323$

CLASS XI **Series 5**

ACE YOUR WAY CBSE



Permutations and Combinations | Binomial Theorem

HIGHLIGHTS

Permutations and Combinations

Fundamental Principle of Counting

Multiplication Rule

If a work is done only when all of the number of works are done, then number of ways of doing that work is equal to the product of number of ways of doing separate works.

Additional Rule

If a work is done only when any one of the number of works is done, then number of ways of doing that work is equal to the sum of number of ways of doing separate works.

Factorial- Factorial is the continued product of first n natural numbers, where n is a positive integer. It is denoted by $n!$ or $\underline{|n}$.

Remark :

- $0! = 1$
- $n! = n(n-1)(n-2)..........3\cdot2\cdot1$

PERMUTATION

The arrangement in definite order of a number of things or objects taken some or all at a time is called permutation.

FORMULAE USED TO FIND THE NUMBER OF PERMUTATIONS IN DIFFERENT SITUATION

Situations		Formula used
When r things are taken out from n different things at a time	If repetition is not allowed	${}^nP_r$ *i.e.*, $\dfrac{n!}{(n-r)!}$
	If repetition is allowed	n^r
When all the things are taken out from n different things at a time		${}^nP_n$ *i.e.*, $n!$
Out of n objects, p are of same kind and the rest are all different		$\dfrac{n!}{p!}$
Out of n objects, p_1 are of first kind, p_2 are of second kind, ..., p_k are of k^{th} kind and rest are all different		$\dfrac{n!}{p_1!p_2!\ldots p_k!}$

COMBINATION

Each of the different groups or selections which can be made by taking some or all of a number of given objects at a time is called combination.

- Number of combinations of n different things taken r at a time is denoted by ${}^nC_r$ and is defined by

$${}^nC_r = \frac{n!}{(n-r)!r!}, \text{ where } 0 \le r \le n.$$

Note:

- For $0 \le r \le n$, ${}^nC_r = {}^nC_{n-r}$
- For $1 \le r \le n$, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- ${}^nP_r = {}^nC_r \cdot r!$, $0 < r \le n$
- If ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x = n - y$

Binomial Theorem

The formula by which any power of a binomial expression can be expanded in the form of a series is known as Binomial Theorem.

Some Expansions :

- $(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}\cdot b + {}^nC_2a^{n-2}\cdot b^2 + ... + {}^nC_nb^n$
- $(a-b)^n = {}^nC_0a^n - {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 - .... +(-1)^n\, {}^nC_nb^n$
- $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$
- $(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 - .... + (-1)^n\, {}^nC_nx^n$

Note:

- The coefficient ${}^nC_r$ occurring in the binomial theorem are known as binomial coefficients.
- The number of terms in the binomial expansion is one more than the index n.

General Term: In the expansion of $(a+b)^n$,

General term $= T_{r+1}$ *i.e.*, $(r+1)^{\text{th}}$ term $= {}^nC_r\, a^{n-r}\cdot b^r$

Middle Term: In the expansion of $(a+b)^n$, the middle term is given by

- $\left(\frac{n}{2}+1\right)^{\text{th}}$ term, if n is even
- $\left(\frac{n+1}{2}\right)^{\text{th}}$ term and $\left(\frac{n+1}{2}+1\right)^{\text{th}}$ term, if n is odd

PROBLEMS

Very Short Answer Type

1. In how many ways can 5 different balls be distributed among three boxes?
2. Find n, if ${}^{n-1}P_3 : {}^nP_4 = 1 : 9$.
3. Prove that
$$\frac{(2n+1)!}{n!} = 2^n\left[1\cdot3\cdot5.....(2n-1)(2n+1)\right]$$
4. A sports team of 11 students is to be constituted by choosing at least 5 from class XI and at least 5 from class XII. If there as 25 students in each of these classes, in how many ways can the teams be constituted?
5. If the 21^{st} and 22^{nd} terms in the expansion of $(1+x)^{44}$ are equal, then find the value of x.

Long Answer Type - I

6. The first three terms of a binomial expansion are 1, 10 and 40. Find the expansion.
7. In how many ways 10 Indians, 5 Americans and 4 Englishmen can be seated in a row so that neither Americans nor Englishmen sit between Indians.
8. On a new year day every student of a class sends a card to every other student. The postman delivers 600 cards. How may students are there in the class?
9. Find the value of the expression ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$.
10. How many different words can be formed with five given letters of which three are vowels and two are consonants, no two vowels being together in any word?

Long Answer Type - II

11. Find the total number of ways of selecting five letters from the letters of the word 'INDEPENDENT'.
12. A group consists of 4 girls and 7 boys. In how many ways can a team of 5 members be selected if the team has
(i) No girls?
(ii) At least one boy and one girl?
(iii) At least three girls?
13. Find the term independent of x in $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$.
14. There are 10 points in a plane, no three of which are in the same straight line, except 4 points, which are collinear. Find

(i) the number of lines obtained from the pairs of these points;

(ii) the number of triangles that can be formed with vertices as these points.
15. If a,b,c be the three consecutive coefficients in the expansion of a power of $(1+x)$, prove that the index of the power is $\frac{2ac+b(a+c)}{b^2-ac}$.

SOLUTIONS

1. Each ball can be put into any one of the three boxes in 3 ways.
$\therefore$ By the multiplication rule of fundamental theorem of counting, the total number of ways of distributing 5 different balls among the three boxes $= 3\times3\times3\times3\times3 = 3^5 = 243$.

2. Given, $\frac{{}^{n-1}P_3}{{}^nP_4} = \frac{1}{9}$
$$\Rightarrow \frac{(n-1)(n-2)(n-3)}{n(n-1)(n-2)(n-3)} = \frac{1}{9}$$
$\Rightarrow\ n = 9$

3. L.H.S. $= \dfrac{(2n+1)!}{n!}$

$$= \frac{(2n+1)\cdot 2n\cdot(2n-1)...[2n-(n-1)]\ .n(n-1)(n-2)...3\cdot 2\cdot 1}{n!}$$

$$= \frac{[(2n+1)(2n-1)...3\cdot 1][2n\cdot 2(n-1)\cdot 2(n-2)...4\cdot 2]}{n!}$$

$$= \frac{[(2n+1)(2n-1)...3.1][2^n\cdot n(n-1)(n-2)...1]}{n!}$$

$$= 2^n[1.3.5...(2n-1)(2n+1)] = \text{R.H.S.}$$

4. A team of 11 students can be constituted as follows :

(i) 5 students from class XI and 6 students from class XII can be selected in ${}^{25}C_5 \times {}^{25}C_6$ ways

(ii) 6 students from class XI and 5 students from class XII can be selected in ${}^{25}C_6 \times {}^{25}C_5$ ways

$\therefore$ Required number of ways

$$= {}^{25}C_5 \times {}^{25}C_6 + {}^{25}C_6 \times {}^{25}C_5$$
$$= 2\cdot {}^{25}C_5 \times {}^{25}C_6$$

5. In the expansion of $(1 + x)^{44}$, $(r + 1)^{\text{th}}$ term is given by $T_{r+1} = {}^{44}C_r x^r$

$\therefore\ T_{21} = {}^{44}C_{20}\, x^{20}$ and $T_{22} = {}^{44}C_{21}\, x^{21}$

Given, $T_{21} = T_{22} \Rightarrow {}^{44}C_{20}\, x^{20} = {}^{44}C_{21}\, x^{21}$

$$\Rightarrow \frac{44!}{20!24!} = \frac{44!}{21!23!}x \Rightarrow x = \frac{21!23!}{20!24!} = \frac{21}{24} = \frac{7}{8}$$

6. Let the binomial expansion be $(a + b)^n$.

Now, $(a + b)^n = a + na^nb + \dfrac{n(n-1)a^{n-1}b^2}{2!} + ...$

According to question, we have

$a = 1$ ...(i)

$na^nb = 10 \Rightarrow nb = 10$ ...(ii)

$\dfrac{n(n-1)}{2!}a^{n-1}b^2 = 40 \Rightarrow \dfrac{n(n-1)}{2!}b^2 = 40$ ...(iii)

Dividing (iii) by the square of (ii), we get

$$\frac{n(n-1)}{2!}b^2 \times \frac{1}{(nb)^2} = \frac{40}{10^2}$$

$$\Rightarrow \frac{n(n-1)}{2!} \times \frac{1}{n^2} = \frac{40}{100}$$

$$\Rightarrow \frac{(n-1)}{2n} = \frac{4}{10} \Rightarrow 5n - 5 = 4n \Rightarrow n = 5$$

From (ii), $5b = 10 \Rightarrow b = 2$

Hence, the binomial expansion is $(1 + 2)^5$.

7. Since neither Americans nor Englishmen should sit between Indians, therefore all the 10 Indians must sit together. Regarding 10 Indians as one person, we have only 1+ 4 + 5 = 10 persons.

These 10 persons can be arranged in a row in 10! ways.

But 10 Indians can be arranged among themselves in 10! ways.

$\therefore$ Required number of ways $= 10! \times 10! = (10!)^2$

8. Let n be the number of students in the class.

Now number of ways in which two students can be selected out of n students $= {}^nC_2$

$\therefore$ Number of pairs of students $= {}^nC_2$

But for each pair of students, number of cards sent is 2 (since if there are two students A and B, A will send a card to B and B will send a card to A.)

$\therefore$ For ${}^nC_2$ pairs, number of cards sent $= 2\cdot {}^nC_2$

According to the question, $2\cdot {}^nC_2 = 600$

$$\Rightarrow\quad 2\cdot\frac{n(n-1)}{2!} = 600 \Rightarrow n^2 - n - 600 = 0$$

$\Rightarrow\quad (n - 25)(n + 24) = 0 \Rightarrow n = 25, -24$

But $n \neq -24$. Therefore, $n = 25$

9. Given expression ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$

$= {}^{47}C_4 + ({}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3)$

$= {}^{47}C_4 + ({}^{47}C_3 + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3)$

$= ({}^{47}C_4 + {}^{47}C_3) + ({}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3)$

$= ({}^{48}C_4 + {}^{48}C_3) + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$

$[\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$

$= ({}^{49}C_4 + {}^{49}C_3) + ({}^{50}C_3 + {}^{51}C_3)$

$= ({}^{50}C_4 + {}^{50}C_3) + {}^{51}C_3 = {}^{51}C_4 + {}^{51}C_3 = {}^{52}C_4$.

10. Since, there is no restriction on consonants, therefore, first of all we arrange the two consonants.

Two consonants can be arranged in 2! ways.

Now if the vowels are put at the places (including the two ends) indicated by '×' then no two vowels will come together.

×	consonant	×	consonant	×

There are three places for three vowels and hence the three vowels can be arranged in these three places in ${}^3P_3 = 3!$ ways.

Hence number of words when no two vowels are together $= 2!\cdot 3! = 12$.

11. Total number of letters = 11

E occurs thrice, N occurs thrice, D occurs twice.

Different letters are I, N, D, E, P, T (six)

Case I. When three letters are identical and remaining two are identical. Letters selection will be :

(i) Three E's and two N's

(ii) Three E's and two D's

(iii) Three N's and two E's

(iv) Three N's and two D's

Number of selection in this case

$= 1 \times 1 + 1 \times 1 + 1 \times 1 + 1 \times 1 = 4$

Case II. When three letters are identical and remaining two are different. Letters selection will be :

(i) Three E's and two out of I, N, D, P, T

(ii) Three N's and two out of I, E, D, P, T

Number of selection in this case $= 1 \times {}^5C_2 + 1 \times {}^5C_2$
$= 20.$

Case III. When two letters are identical of one type, two are identical of second type and rest one is different. Letters selection will be :

(i) Two E's two N's and one out of I,D,P,T

(ii) Two E's two D's and one out of I,N,P,T

(iii) Two N's two D's and one out of I,E,P,T

Number of selection in this case

$= (1 \times 1 \times {}^4C_1 + 1 \times 1 \times {}^4C_1 + 1 \times 1 \times {}^4C_1) = 12.$

Case IV. When two letters are identical and remaining three are different. Letters selection will be :

(i) Two E's and three out of I, N, D, P, T

(ii) Two N's and three out of I, E, P, D, T

(iii) Two D's and three out of of I, E, P, N, T

$\therefore$ Number of selection in this case

$= 1 \times {}^5C_3 + 1 \times {}^5C_3 + 1 \times {}^5C_3 = 30.$

Case V. When all the 5 letters are different.

Number of selection $= {}^6C_5 = 6$

$\therefore$ Required number of ways $= 4+20+12+30+6 = 72.$

12. Number of girls = 4, number of boys = 7

(i) Since the team has no girls, therefore, 5 members must be selected from 7 boys.

$\therefore$ Required number of ways $= {}^7C_5 = {}^7C_2 = \dfrac{7\times 6}{2!} = 21.$

(ii) The team has atleast one boy and one girl, therefore the team selection will be following.

No. of boys selected	No. of girls selected	No. of ways
1	4	${}^7C_1 . {}^4C_4 = 7$
2	3	${}^7C_2 . {}^4C_3 = 84$
3	2	${}^7C_3 . {}^4C_2 = 210$
4	1	${}^7C_4 . {}^4C_1 = 140$

$\therefore$ Required number of ways = 7 + 84 + 210 + 140
= 441.

(iii) Required number of ways $= {}^4C_3 . {}^7C_2 + {}^4C_4 . {}^7C_1$
$= 84 + 7 = 91.$

13. Let r^{th} term be independent of x.

Now, in the expansion of $\left(\dfrac{3}{2}x^2 - \dfrac{1}{3x}\right)^9$, r^{th} term is given by,

$$T_r = {}^9C_{r-1}\left(\frac{3}{2}x^2\right)^{9-r+1}\left(-\frac{1}{3x}\right)^{r-1}$$

$$= {}^9C_{r-1}\left(\frac{3}{2}\right)^{10-r} . \left(x^2\right)^{10-r}\left(-\frac{1}{3}\right)^{r-1}\left(\frac{1}{x}\right)^{r-1}$$

$$= (-1)^{r-1} \times {}^9C_{r-1}\left(\frac{3}{2}\right)^{10-r}\frac{1}{3^{r-1}}x^{21-3r} \qquad \text{...(i)}$$

Since r^{th} term is independent of x,

$\therefore\ 21 - 3r = 0 \Rightarrow r = 7$

From (i), $T_7 = (-1)^6 . {}^9C_6\left(\dfrac{3}{2}\right)^{10-7} . \dfrac{1}{3^6}$

$$= {}^9C_6\frac{3^3}{2^3}\cdot\frac{1}{3^6} = \frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3}\cdot\frac{1}{8\cdot 3^3} = \frac{7}{18}$$

Hence, term independent of $x = \dfrac{7}{18}$

14. (i) Number of lines formed by joining the 10 points, taking 2 at a time $= {}^{10}C_2 = \left(\dfrac{10\times 9}{2\times 1}\right) = 45.$

Number of lines formed by joining the 4 points, taking 2 at a time $= {}^4C_2 = \left(\dfrac{4\times 3}{2\times 1}\right) = 6.$

But, 4 collinear points, when joined pair wise, give 1 line.

$\therefore$ The required number of straight lines

$= (45 - 6 + 1) = 40.$

(ii) Number of triangles formed by joining the 10 points, taking 3 at time

$$= {}^{10}C_3 = \left(\frac{10\times9\times8}{3\times2\times1}\right) = 120.$$

Also, the number of triangles formed by joining the 4 points, taking 3 at a time $= {}^4C_3 = {}^4C_1 = 4$.

But, there is no triangle formed by joining any 3 points out of the 4 collinear points.

$\therefore$ The required number of triangle formed

$= (120 - 4) = 116.$

15. Let the index of the power be n whose value is to be obtained. Let a, b, c be the r^{th}, $(r + 1)^{\text{th}}$ and $(r + 2)^{\text{th}}$ coefficients respectively in the expansion of $(1 + x)^n$.

Then, $a = {}^nC_{r-1}$ ...(i)

$b = {}^nC_r$ ...(ii)

$c = {}^nC_{r+1}$ ...(iii)

Dividing (i) by (ii), we get

$$\frac{a}{b} = \frac{{}^nC_{r-1}}{{}^nC_r} = \frac{n!}{(r-1)!(n-r+1)!} \times \frac{r!(n-r)!}{n!}$$

$$\Rightarrow \frac{a}{b} = \frac{r}{n-r+1} \Rightarrow an - ar + a = br$$

$$\Rightarrow an + a = r(a + b) \quad ...(1)$$

Now, dividing (ii) by (iii), we get

$$\frac{b}{c} = \frac{{}^nC_r}{{}^nC_{r+1}} = \frac{n!}{r!(n-r)!} \times \frac{(r+1)!(n-r-1)!}{n!}$$

$$\Rightarrow \frac{b}{c} = \frac{r+1}{n-r} \Rightarrow bn - br = cr + c$$

$$\Rightarrow bn - c = r(b + c) \quad ...(2)$$

Now, eliminating r from (1) and (2), we get

$$(an + a) = \frac{(bn - c)(a + b)}{(b + c)}$$

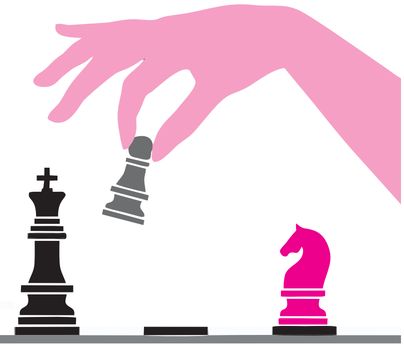
$$\Rightarrow abn + acn + ab + ac = abn + b^2n - ac - bc$$

$$\Rightarrow 2ac + ab + bc = b^2n - acn$$

$$\Rightarrow n = \frac{2ac + b(a + c)}{b^2 - ac}$$

◈◈

Challenging PROBLEMS ON TRIGONOMETRY



SECTION-I

Single Correct Answer Type

1. $(1 - \cot 22°)(1 - \cot 23°) =$

(a) 1 (b) 2 (c) 3 (d) 4

2. The expression

$\sqrt{\sin^4 x + 4\cos^2 x} - \sqrt{\cos^4 x + 4\sin^2 x}$ is equal to

(a) 3 (b) –3
(c) $\cos 2x$ (d) $-\cos 2x$

3. The values of the parameter θ for which the expression $\dfrac{\tan(x-\theta)+\tan x+\tan(x+\theta)}{\tan(x-\theta)\cdot\tan x\cdot\tan(x+\theta)}$ is independent of x is

(a) $\pm\dfrac{\pi}{3}+n\pi$ (b) $\pm\dfrac{\pi}{2}+n\pi$
(c) $(2n+1)\pi$ (d) $n\pi+\sin^{-1}\left(\dfrac{1}{3}\right)$

4. Let $a_1, a_2, \ldots$ be real constants and

$$y(x)=\cos(a_1+x)+\frac{1}{2}\cos(a_2+x)+\frac{1}{2^2}\cos(a_3+x)+\ldots+\frac{1}{2^{n-1}}\cos(a_n+x)$$

If x_1, x_2 are real and $y(x_1) = y(x_2) = 0$ then $x_2 - x_1 =$

(a) $n\pi,\ n \in I$ (b) $\dfrac{n\pi}{2},\ n \in I$
(c) $\dfrac{n\pi}{3},\ n \in I$ (d) $\dfrac{n\pi}{4},\ n \in I$

5. Let $a, b, c, d \in [0, \pi]$ and $\sin a + 7\sin b = 4(\sin c + 2\sin d)$ and $\cos a + 7\cos b = 4(\cos c + 2\cos d)$ then $\dfrac{\cos(a-d)}{\cos(b-c)} =$

(a) $\dfrac{2}{7}$ (b) $\dfrac{7}{2}$ (c) $\dfrac{4}{7}$ (d) $\dfrac{7}{4}$

6. The number of values of $\alpha \in [0, 2\pi]$ for which the three element set $A = \{\sin\alpha, \sin 2\alpha, \sin 3\alpha\}$ and $B = \{\cos\alpha, \cos 2\alpha, \cos 3\alpha\}$ are equal is

(a) 0 (b) 2 (c) 4 (d) 6

7. For $x \in R$, the minimum value of $|\sin x + \cos x + \tan x + \sec x + \operatorname{cosec} x + \cot x|$ is

(a) $2\sqrt{2}$ (b) $2\sqrt{2}-1$
(c) $2\sqrt{2}+1$ (d) 2

8. The number of positive integers n for which the equality $\dfrac{\sin(n\alpha)}{\sin\alpha}-\dfrac{\cos(n\alpha)}{\cos\alpha}=(n-1)$ holds true for all $\alpha \neq \dfrac{k\pi}{2},\ k \in I$

(a) zero (b) 1
(c) 2 (d) infinite

9. Let $\alpha, \beta, \gamma, \delta \in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$ be real numbers such that $\sin\alpha + \sin\beta + \sin\gamma + \sin\delta = 1$ and $\cos 2\alpha + \cos 2\beta + \cos 2\gamma + \cos 2\delta \geq \dfrac{10}{3}$, then $\alpha \in$

(a) $[0, \pi/6]$ (b) $[0, \pi/3]$
(c) $[0, \pi/4]$ (d) $[0, \pi/2]$

SECTION-II

Multiple Correct Answer Type

10. In ΔABC, $3\sin A + 4\cos B = 6$ and $4\sin B + 3\cos A = 1$ then angle C is

(a) 30° (b) 60° (c) 90° (d) 150°

11. If for $A \geq 0, B \geq 0, A + B = 60°$ and $y = \tan A \cdot \tan B$ then

(a) $y_{max} = 3$ (b) $y_{min} = \dfrac{1}{3}$
(c) $y_{max} = \dfrac{1}{3}$ (d) $y_{min} = 0$

12. If $\sin x + \cos x + \tan x + \sec x + \operatorname{cosec} x + \cot x = 7$ and $\sin 2x = a + b\sqrt{7}$ then

(a) $a = 22$ (b) $a = -22$
(c) $b = 8$ (d) $b = -8$

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13. Let $P_n(U)$ be a polynomial in U of degree n. Then for every positive integer n, $\sin(2nx)$ is expressible in

(a) $P_{2n}(\sin x)$ (b) $P_{2n}(\cos x)$

(c) $\cos x \cdot P_{2n-1}(\sin x)$ (d) $\sin x \cdot P_{2n-1}(\cos x)$

14. Solutions for the equation $\cos^2 x + \cos^2 2x + \cos^2 3x = 1$ are of the form

(a) $\frac{\pi}{2} + n\pi, n \in I$ (b) $\frac{\pi}{4} + \frac{n\pi}{2}, n \in I$

(c) $\frac{\pi}{6} + \frac{n\pi}{3}, n \in I$ (d) $n\pi + \frac{\pi}{5},\ n \in I$

SECTION-III

Comprehension Type

Paragraph for Question No. 15 and 16

Let $ABCD$ .... JKL be a regular dodecagon and let R be the circumradius.

15. $\frac{AB}{AF} + \frac{AF}{AB} =$

(a) 1 (b) $\sqrt{2}$ (c) 3 (d) 4

16. $AB^2 + AC^2 + AD^2 + AE^2 + AF^2 =$

(a) R^2 (b) $5R^2$

(c) $10R^2$ (d) $16R^2$

Paragraph for Question No. 17 and 18

Given that $\frac{\cos^4\alpha}{\cos^2\beta} + \frac{\sin^4\alpha}{\sin^2\beta} = 1.$

17. $\frac{\cos^2\alpha}{\cos^2\beta} + \frac{\cos^2\beta}{\cos^2\alpha} =$

(a) 2 (b) 4 (c) 8 (d) 16

18. $\frac{\cos^4\beta}{\cos^2\alpha} + \frac{\sin^4\beta}{\sin^2\alpha} =$

(a) 1 (b) 2 (c) 4 (d) 8

SECTION-IV

Integer Answer Type

19. Let A_1A_2 .... A_{14} be a regular polygon with 14 sides inscribed in a circle of radius R, then

$(A_1A_3)^2 + (A_1A_7)^2 + (A_3A_7)^2 = \lambda R^2$ for $\lambda =$ ______

20. The number of real distinct solutions to the system of equations $x^3 - 3x = y$, $y^3 - 3y = z$ and $z^3 - 3z = x$ is λ. Find $\frac{\lambda}{3}$

21. If $\frac{\tan 1}{\cos 2} + \frac{\tan 2}{\cos 4} + \ldots + \frac{\tan 1024}{\cos 2048} = \tan\lambda - \tan\mu$ then $\left[\frac{\lambda - \mu}{1000}\right] =$ ______, $[\cdot]$ denotes the greatest integer function.

22. The area of the region contained by all the points (x, y) such that $x^2 + y^2 \le 100$ and $\sin(x + y) \ge 0$ is $10\,k\pi$ for $k =$ ______

23. If $\alpha, \beta \ne \frac{n\pi}{2}, n \in I,$ then minimum value of $\frac{\sec^4\alpha}{\tan^2\beta} + \frac{\sec^4\beta}{\tan^2\alpha}$ is

24. The sum of all $x \in [0, 2\pi]$ such that $3\cot^2 x + 8\cot x + 3 = 0$ is $k\pi$ for $k =$ ______

25. If $\cot\left(7\frac{1}{2}\right)^\circ = \sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d}$ then $a + b + c + d - 8 =$ ______

SOLUTIONS

1. **(b) :** Using $\cot(22° + 23°) = \cot 45° = 1$, we have $\frac{\cot 22° \cot 23° - 1}{\cot 22° + \cot 23°} = 1,$ simplifying, we have $(1 - \cot 22°)(1 - \cot 23°) = 2.$

2. **(c) :** $\sin^4 x + 4\cos^2 x = (2 - \sin^2 x)^2$ and $\cos^4 x + 4\sin^2 x = (2 - \cos^2 x)^2$

So, given expression $= |2 - \sin^2 x| - |2 - \cos^2 x|$
$= (2 - \sin^2 x) - (2 - \cos^2 x) = \cos 2x$

3. **(a) :** The given expression is simplified to $\frac{4\cos 2\theta + 2}{\cos 2\theta + \cos 2x} - 3$, which is independent of x

iff $\cos 2\theta = -\frac{1}{2}$. Hence $\theta = \pm\frac{\pi}{3} + n\pi,\ n \in I.$

4. **(a) :** Using $\cos(\theta + \phi) = \cos\theta \cos\phi - \sin\theta \sin\phi$ in each term and simplifying, we have $y(x) = \lambda\sin(x + \alpha)$ for some constants λ and α. So, zeroes of $y(x)$ are of the form $x + \alpha = n\pi$, $(n \in I)$ and so, $y(x_1) = y(x_2)$
$\Rightarrow x_1 - x_2 = n\pi$

5. **(b) :** Rewriting the given two equations as
$\sin a - 8\sin d = 4\sin c - 7\sin b$
and $\cos a - 8\cos d = 4\cos c - 7\cos b$
and then squaring and adding, we have
$65 - 16\cos(a - d) = 65 - 56\cos(b - c)$
$\Rightarrow \frac{\cos(a-d)}{\cos(b-c)} = \frac{7}{2}.$

6. (c) : Since, $A = B$, so, sum of the elements in each set are equal hence,

$\sin\alpha + \sin2\alpha + \sin3\alpha = \cos\alpha + \cos2\alpha + \cos3\alpha$

Simplifying, we get, $\sin2\alpha = \cos2\alpha$

Hence, $\alpha = \frac{\pi}{8} + \frac{n\pi}{2}$

So, in $[0, 2\pi]$, total 4 possible values of α.

7. (b) : Putting $a = \sin x$, $b = \cos x$ and $c = \sin x + \cos x$, the given expression simplifies to $\left|c + \frac{2}{c-1}\right|$

$= \left|c - 1 + \frac{2}{c-1} + 1\right| \geq |-2\sqrt{2} + 1|$ using A.M. ≥ G.M.

So, required minimum value is $(2\sqrt{2} - 1)$.

8. (c) : The given expression simplifies to

$\sin(n-1)\alpha = \frac{(n-1)}{2}\sin2\alpha$

For $n \geq 4$ and putting $\alpha = \pi/4$, we have

$\sin\left((n-1)\frac{\pi}{4}\right) = \frac{n-1}{2} \geq \frac{3}{2}$, which is not possible.

For $n = 2$, $\sin\alpha = \frac{\sin2\alpha}{2}$ which is not true for $\alpha = \pi/4$.

Hence, only possible values for n are 1 and 3.

9. (a) : Let $a = \sin\alpha$, $b = \sin\beta$, $c = \sin\gamma$ and $d = \sin\delta$ then from the given equations, we have

$a + b + c + d = 1$ and

$a^2 + b^2 + c^2 + d^2 \leq \frac{1}{3}$.

Applying R.M.S ≥ A.M. on (a, b, c) we have

$\frac{1}{3} \geq a^2 + b^2 + c^2 + d^2 \geq \frac{(a+b+c)^2}{3} + d^2 = \frac{(1-d)^2}{3} + d^2$

i.e., $2d^2 - d \leq 0$ *i.e.*, $d \in \left[0, \frac{1}{2}\right]$

Hence, $\delta \in \left[0, \frac{\pi}{6}\right]$

Similarly $\alpha, \beta, \gamma \in \left[0, \frac{\pi}{6}\right]$

10. (a) : Squaring and adding the two given equations, we have,

$\sin(A+B) = \frac{1}{2}$. So, $C = 30°$ or $150°$

Note that $C = 150°$ is not possible. Hence $C = 30°$.

11. (c, d) : Eliminating B, we have

$\tan^2 A + \sqrt{3}(y-1)\tan A + y = 0$

and $D \geq 0$ gives $y \leq \frac{1}{3}$ or $y \geq 3$

But each of $\tan A$ and $\tan B$ is less than $\sqrt{3}$. Hence, $y \geq 3$ is not possible. Hence, $y \leq \frac{1}{3}$ and $\tan A \cdot \tan B$ is non-negative.

12. (a, d) : $(\sin x + \cos x)\left(1 + \frac{1}{\sin x \cos x}\right) = 7 - \frac{2}{\sin 2x}$

Squaring and simplifying, we have

$\sin^3 2x - 44\sin^2 2x + 36\sin 2x = 0$

Hence, $\sin 2x = 22 - 8\sqrt{7}$.

13. (c, d) : Notice that

$\sin4x = 2\sin x \cos x(2\cos^2 x - 1)$

$= \sin x \cdot P_3(\cos x)$ or $\cos x \cdot P_3(\sin x)$

So, $\sin(2nx) = 2\sin nx \cdot \cos nx$

$= \sin nx \cdot P_{2n-1}(\cos x)$ or $\cos x \cdot P_{2n-1}(\sin x)$

14. (a, b, c) : Using

$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

We have, $\cos2x + \cos4x + 2\cos^2 3x = 0$

i.e. $4\cos3x \cos2x \cos x = 0$

Hence, $x = \left\{\frac{\pi}{2} + n\pi, \frac{\pi}{4} + \frac{n\pi}{2}, \frac{\pi}{6} + \frac{n\pi}{3}\right\}$

(15-16):

15. (d) **16. (c)**

Notice that $AB = 2R\sin\frac{\pi}{12}$ and $AF = 2R\sin\frac{5\pi}{12}$ and

and $\sin\left(\frac{5\pi}{12}\right) = \cos\left(\frac{\pi}{2} - \frac{5\pi}{12}\right) = \cos\frac{\pi}{12}$

(17-18):

17. (a) **18. (a)**

Use the identity $\frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}$ and equality holds for $ay = bx$.

Now, put $a = \cos^2\alpha$, $b = \sin^2\alpha$, $x = \cos^2\beta$, $y = \sin^2\beta$ to get the various results.

19. (7) : $A_1A_3 = 2R\sin(\pi/7)$, $A_3A_7 = 2R\sin(2\pi/7)$, $A_1A_7 = 2R\sin(3\pi/7)$

The given expression

$= 2R^2\left[3 - \left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right)\right]$

and $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$

$= \frac{1}{2\sin\frac{2\pi}{7}}\left(\sin\frac{4\pi}{7} + \sin\frac{6\pi}{7} - \sin\frac{2\pi}{7} + \sin\frac{8\pi}{7} - \sin\frac{4\pi}{7}\right)$

$= -\frac{1}{2}$. So, the given expression $= 7R^2$.

20. (9) : Putting, $x = 2\cos\alpha$, $y = 2\cos\beta$ and $z = 2\cos\gamma$ for $\alpha, \beta, \gamma \in [0, \pi]$ the given system of equations become

$$2\cos3\alpha = 2\cos\beta,\ 2\cos3\beta = 2\cos\gamma,\ 2\cos3\gamma = 2\cos\alpha$$

Hence, $\cos27\alpha = \cos\alpha$

i.e. 27 different values.

$\therefore\ \lambda = 27$. So, $\frac{\lambda}{3} = 9$

21. (2) : Notice that

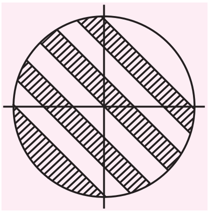
$$\frac{\tan\theta}{\cos2\theta} = \frac{\tan\theta(1+\tan^2\theta)}{1-\tan^2\theta} = \tan2\theta - \tan\theta$$

Hence, the given L.H.S is

$$(\tan2 - \tan1) + (\tan4 - \tan2) + \ldots + (\tan2048 - \tan1024)$$

$$= \tan2048 - \tan1$$

22. (5) : Using the symmetricity of the diagram, required area of the region is



$$\frac{1}{2}\times100\pi$$

i.e., 50π sq. units

23. (8) : Setting $a = \tan^2\alpha$ and $b = \tan^2\beta$, we have,

$$\frac{\sec^4\alpha}{\tan^2\beta} + \frac{\sec^4\beta}{\tan^2\alpha} = \frac{(a+1)^2}{b} + \frac{(b+1)^2}{a}$$

$$= \left(\frac{a^2}{b} + \frac{b^2}{a} + \frac{1}{a} + \frac{1}{b}\right) + 2\left(\frac{a}{b} + \frac{b}{a}\right)$$

$$\geq 4\times\sqrt[4]{\frac{a^2}{b}\cdot\frac{b^2}{a}\cdot\frac{1}{a}\cdot\frac{1}{b}} + 2\times2\sqrt{\frac{a}{b}\cdot\frac{b}{a}}$$

$$= 8$$

24. (5) : Let $t = \cot x$ then $3t^2 + 8t + 3 = 0$ has roots

$$t_1 = \frac{-4+\sqrt{7}}{3} \text{ and } t_2 = \frac{-4-\sqrt{7}}{3}$$

$t_1 = \cot x = \frac{-4+\sqrt{7}}{3}$ has roots whose sum $= \frac{3\pi}{2}$

[by using graph of $\cot x$]

and $t_2 = \cot x = \frac{-4-\sqrt{7}}{3}$ has roots whose sum $= \frac{7\pi}{2}$

Hence, total sum $= 5\pi$

25. (7) : Use, $\tan\left(\frac{A}{2}\right) = \frac{-1\pm\sqrt{1+\tan^2 A}}{\tan A}$

and put $A = 15°$ to get

$$\cot\left(7\frac{1}{2}^\circ\right) = \sqrt{6}+\sqrt{4}+\sqrt{3}+\sqrt{2}$$

Science Grads Trump Engineers as Data Analytics Gains Currency

Cos prefer science, economics grads for their number crunching abilities for analytics roles

With the rise of analytics and big data, demand for science - especially from streams such as Physics, Chemistry, Mathematics, Statistics - and Economics graduates is booming. Be it IT services companies or Big Four consultancies providing such services, plain vanilla science graduates and economics students are being preferred over engineers for their number crunching abilities for analytics roles.

According to IT staffing firm Teamlease, fresher level salaries for economics or statistics graduates range from ₹ 4-7 lakh per year, compared with ₹ 3.5 lakh for engineering graduates for analytics jobs. At the middle level, pay cheques range from ₹ 12-15 lakh per year. "We are seeing more BSc graduates being preferred for data analytics roles because they have those required skills. Hiring is more off-campus than on campus," said Sangeeta Gupta, senior vice-president at industry body Nasscom.

An internal survey by Nasscom last year revealed that, excluding BPOs, about 5-6% of the industry headcount was from non-engineering fields. "And that percentage is growing," said Gupta. Among the top IT firms, Tata Consultancy Services and Wipro run programmes exclusively to hire and train BSc and MSc graduates. The rising demand for data professionals can be gauged by the fact that over 1,00,000 new positions for data analysts and researchers are expected to be created by next year, according to Alka Dhingra, assistant general manager at Teamlease.

The demand for such graduates is driven by the pace at which big data and analytics is booming.

While BSc candidates are preferred for more entry-level jobs, the middle and higher level hires working in the field are increasingly coming from statistics and economics backgrounds.

What's even more interesting is that companies are also looking at BSc graduates for roles such as infrastructure management, squeezing out the engineering freshers who typically held these roles. At this level, the BSc hires are cheaper, and get paid in the ₹ 1.8-2 lakh range. "The industry is moving to more skill-based hiring, rather than just engineers for everything," said Nasscom's Gupta.

Another reason why science graduates are preferred is because they possess the "right mindset for learning on the job skills, they don't drop out as often, are not finicky about what location they get," said an industry source.

Courtesy : The Economic Times

MPP-3 MONTHLY Practice Problems

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer.

Self check table given at the end will help you to check your readiness.

Trigonometric Functions, Mathematical Induction & Mathematical Reasoning

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. If $\theta = \frac{\pi}{4n}$, then value of $\tan\theta \tan 2\theta \ldots \tan(2n-1)\theta$ equals

(a) -1 (b) 1 (c) 0 (d) 2

2. If $U_n = \sin n\theta \sec^n\theta$, $V_n = \cos n\theta \sec^n\theta \neq 1$, then $\frac{V_n - V_{n-1}}{U_{n-1}} + \frac{1}{n}\frac{U_n}{V_n}$ is equal to

(a) 0 (b) $\tan\theta$

(c) $-\tan\theta + \frac{\tan n\theta}{n}$ (d) $\tan\theta + \frac{\tan n\theta}{n}$

3. The value of $e^{\log_{10}\tan 1^\circ + \log_{10}\tan 2^\circ + \log_{10}\tan 3^\circ + \ldots + \log_{10}\tan 89^\circ}$ is equal to

(a) 0 (b) e

(c) $1/e$ (d) none of these

4. Let $P(n)$ be the statement $n^3 + n$ is $3m$ such that m is a positive integer, then which of the following is true?

(a) $P(1)$ (b) $P(2)$ (c) $P(3)$ (d) $P(4)$

5. In a triangle ABC, let $\angle C = \frac{\pi}{2}$. If r is the inradius and R is the circumradius of the triangle ABC, then $2(r + R)$ equals

(a) $a + b$ (b) $b + c$

(c) $c + a$ (d) $a + b + c$

6. Identify the false statement

(a) $\sim[p \vee (\sim q)] \equiv (\sim p) \wedge q$

(b) $[p \vee q] \vee (\sim p)$ is a tautology

(c) $\sim(p \vee q) \equiv (\sim p) \vee (\sim q)$

(d) $\sim[p \wedge (\sim p)]$ is a tautology

One or More Than One Options Correct Type

7. If $(a - b)\sin(\theta + \phi) = (a + b)\sin(\theta - \phi)$ and $a\tan\frac{\theta}{2} - b\tan\frac{\phi}{2} = c$, than

(a) $b\tan\phi = a\tan\theta$

(b) $a\tan\phi = b\tan\theta$

(c) $\sin\phi = \frac{2bc}{a^2 - b^2 - c^2}$

(d) $\sin\theta = \frac{2ac}{a^2 - b^2 + c^2}$

8. $\left(\frac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\frac{\sin A + \sin B}{\cos A - \cos B}\right)^n$ (n, even or odd) is equal to

(a) $2\tan^n\left(\frac{A-B}{2}\right)$ (b) $2\cot^n\left(\frac{A-B}{2}\right)$

(c) 0 (d) none of these

9. The statements $\sim(p \leftrightarrow q)$ is

(a) Contingency

(b) a tautology

(c) a fallacy

(d) equivalent to $(p \wedge \sim q) \vee (q \wedge \sim p)$

10. If $\frac{\sin^4 x}{2}+\frac{\cos^4 x}{3}=\frac{1}{5}$, then

(a) $\tan^2 x=\frac{2}{3}$ (b) $\frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}=\frac{1}{125}$

(c) $\tan^2 x=\frac{1}{3}$ (d) $\frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}=\frac{2}{125}$

11. There exist a triangle ABC satisfying the conditions,

(a) $b\sin A=a,\ A<\frac{\pi}{2}$

(b) $b\sin A<a,\ A>\frac{\pi}{2}, b=a$

(c) $b\sin A>a,\ A<\frac{\pi}{2}$

(d) $b\sin A<a,\ A<\frac{\pi}{2}, b>a$

12. CF is the internal bisector of angle C of ΔABC, then CF is equal to

(a) $\frac{2ab}{a+b}\cos\frac{C}{2}$ (b) $\frac{a+b}{2ab}\cos\frac{C}{2}$

(c) $\frac{b\sin A}{\sin\left(B+\frac{C}{2}\right)}$ (d) none of these

13. Which of the given below statements is "inclusive or" statement ?

(a) Sun rises or moon rises
(b) All integers are positive or negative
(c) Two lines intersect at a point or are parallel
(d) The school is closed if it is holiday or a sunday.

Comprehension Type

If $P_n=\sin^n\theta+\cos^n\theta$, where $n\in W$ and $\theta\in R$

14. If $P_{n-2}-P_n=\sin^2\theta\cos^2\theta\, P_\lambda$, then the value of λ is

(a) $n-1$ (b) $n-2$ (c) $n-3$ (d) $n-4$

15. The value of $\frac{P_7-P_5}{P_5-P_3}$ is

(a) $\frac{P_7}{P_5}$ (b) $\frac{P_5}{P_3}$ (c) $\frac{P_3}{P_1}$ (d) $\frac{P_3}{P_5}$

Matrix Match Type

16. Match the columns.

	Column I		Column II
(P)	If $\theta+\phi=\frac{\pi}{2}$, where θ and ϕ are positive, then the maximum value of $(\sin\theta+\sin\phi)\ \sin\left(\frac{\pi}{4}\right)$ is	(1)	3
(Q)	If $\sin\theta-\sin\phi=a$ and $\cos\theta+\cos\phi=b$, then minimum value of a^2+b^2 is	(2)	1
(R)	If $3\sin\theta+5\cos\theta=5$, $(\theta\neq 0)$ then the value of $5\sin\theta-3\cos\theta$ is	(3)	0

	P	Q	R
(a)	2	1	3
(b)	3	2	1
(c)	1	2	3
(d)	2	3	1

Integer Answer Type

17. The number of values of x in the interval $[0, 3\pi]$ satisfying the equation $2\sin^2 x+5\sin x-3=0$ is

18. If $A+B+C=\pi$ and $\frac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}$ $=\lambda\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$, then last digit of the value of $1+2\lambda+3\lambda^2+4\lambda^3$ must be

19. The period of $2\sin\left(3\pi x+\frac{\pi}{4}\right)+3|\cos 5\pi x|$ is

20. The number of values of θ in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ such that $\theta\neq\frac{n\pi}{5}$ for $n=0, \pm1, \pm2$ and $\tan\theta=\cot 5\theta$ as well as $\sin 2\theta=\cos 4\theta$ is

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SELF CHECK

No. of questions attempted ……
No. of questions correct ……
Marks scored in percentage ……

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

CONCEPT BOOSTERS

Class XII

Limits, Continuity and Differentiability

This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

DEFINITION

Limit of a function $f(x)$ is said to exist as, $x \to a$ when,

$$\lim_{h \to 0^-} f(a-h) = \operatorname{Lim}_{h \to 0^+} f(a+h)$$

(Left hand limit) (Right hand limit)

Note that we are not interested in knowing about what happens at $x = a$. Also note that if L.H.L. & R.H.L. are both tending towards '∞' or '–∞' then it is said to be infinite limit.

Remember, '$x \to a$' means that x is approaching to 'a' but not equal to 'a'.

THEOREMS ON LIMITS

Let $\operatorname{Lim}_{x \to a} f(x) = l$ and $\operatorname{Lim}_{x \to a} g(x) = m$. If l & m are finite, then

(A) $\operatorname{Lim}_{x \to a} \{f(x) \pm g(x)\} = l \pm m$

(B) $\operatorname{Lim}_{x \to a} \{f(x) \cdot g(x)\} = l \cdot m$

(C) $\operatorname{Lim}_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{l}{m}$, provided $m \neq 0$

(D) $\operatorname{Lim}_{x \to a} kf(x) = k \operatorname{Lim}_{x \to a} f(x) = kl$, where k is a constant.

(E) $\operatorname{Lim}_{x \to a} f\{g(x)\} = f\left(\operatorname{Lim}_{x \to a} g(x)\right) = f(m)$

provided f is continuous at $g(x) = m$.

INDETERMINATE FORMS

If on putting $x = a$ in $f(x)$ any one of $\dfrac{0}{0}, \dfrac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, ∞^0, 0^0, 1^∞ is obtained, then the limit is called an indeterminate form. All these forms are interchangeable, *i.e.* we can change one form to other by suitable substitutions.

In such cases $\operatorname{Lim}_{x \to a} f(x)$ may exist.

REMARK :

(i) $\infty + \infty = \infty$

(ii) $\infty \times \infty = \infty$

(iii) $(a/\infty) = 0$, if a is finite.

(iv) $\dfrac{a}{0}$ is not defined for any $a \in R$.

(v) $\operatorname{Lim}_{x \to 0} \dfrac{x}{x}$ is an indeterminate form whereas

$\operatorname{Lim}_{x \to 0} \dfrac{[x^2]}{x^2}$ is not an indeterminate form.

L'HOSPITAL'S RULE

If $f(x)$ & $g(x)$ be two functions of x such that

(i) $\operatorname{Lt}_{x \to a} f(x) = \operatorname{Lt}_{x \to a} g(x) = 0$

or $\operatorname{Lt}_{x \to a} f(x) = \infty = \operatorname{Lt}_{x \to a} g(x)$

(ii) $f(x)$ and $g(x)$ both continuous at $x = a$

(iii) both differentiable at $x = a$

(iv) $f'(x)$ and $g'(x)$ both continuous at $x = a$ then provided

$$\operatorname{Lt}_{x \to a} \frac{f(x)}{g(x)} = \operatorname{Lt}_{x \to a} \frac{f'(x)}{g'(x)},$$

If $f'(a) = g'(a) = 0$ then,

$$\operatorname{Lt}_{x \to a} \frac{f'(x)}{g'(x)} = \operatorname{Lt}_{x \to a} \frac{f''(x)}{g''(x)}$$

We continue the process till to get the finite number.

STANDARD LIMITS

- $\lim\limits_{x\to a} f(x) = A > 0$ and $\lim\limits_{x\to a} \phi(x) = B$ (a finite quantity) then $\lim\limits_{x\to a} [f(x)]^{\phi(x)} = A^B$
- $\lim\limits_{x\to 0} \frac{\sin x}{x} = \lim\limits_{x\to 0} \frac{\tan x}{x} = \lim\limits_{x\to 0} \frac{\tan^{-1} x}{x}$
$$= \lim\limits_{x\to 0} \frac{\sin^{-1} x}{x} = 1$$
- $\lim\limits_{x\to 0} (1+ax)^{1/x} = e^a$; $\left(\lim\limits_{x\to 0} (1+x)^{1/x} = e\right)$
- $\lim\limits_{x\to \infty} \left(1+\frac{a}{x}\right)^x = e^a$; $\left(\lim\limits_{x\to \infty} \left(1+\frac{1}{x}\right)^x = e\right)$
- $\lim\limits_{x\to 0} \frac{e^x - 1}{x} = 1$; $\lim\limits_{x\to 0} \frac{a^x - 1}{x} = \log_e a,\ a > 0$
- $\lim\limits_{x\to 0} \frac{\ln(1+x)}{x} = 1$
- $\lim\limits_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$

USE OF SUBSTITUTION IN SOLVING LIMIT PROBLEMS

Sometimes in solving limit problems we convert $\lim\limits_{x\to a} f(x)$ into $\lim\limits_{h\to 0} f(a+h)$ or $\lim\limits_{h\to 0} f(a-h)$ according as need of the problems.

Some important limits :

(i) $\lim\limits_{x\to\infty} \frac{\ln x}{x} = 0$ (ii) $\lim\limits_{x\to\infty} \frac{x}{e^x} = 0$

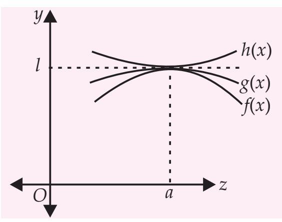
(iii) $\lim\limits_{x\to\infty} \frac{x^n}{e^x} = 0$ (iv) $\lim\limits_{x\to\infty} \frac{(\ln x)^n}{x} = 0$

(v) $\lim\limits_{x\to 0^+} x(\ln x)^n = 0$

(vi) $\lim\limits_{n\to\infty} (1-h)^n = 0$ & $\lim\limits_{n\to\infty} (1+h)^n \to \infty$, where $h \to 0^+$.

SANDWICH THEOREM/SQUEEZE THEOREM:

- If $f(x) \le g(x) \le h(x)\ \forall\, x$ & $\lim\limits_{x\to a} f(x) = l = \lim\limits_{x\to a} h(x)$ then $\lim\limits_{x\to a} g(x) = l$



CONTINUITY

- A function $f(x)$ is said to be continuous at $x = c$, if $\lim\limits_{x\to c} f(x) = f(c)$

  i.e. f is continuous at $x = c$ if

  $\lim\limits_{h\to 0} f(c-h) = \lim\limits_{h\to 0} f(c+h) = f(c)$

TYPES OF DISCONTINUITY

(i) Discontinuity of 1st kind

If both $\lim\limits_{x\to c^-} f(x)$ and $\lim\limits_{x\to c^+} f(x)$ exist finitely then the function f is said to have discontinuity of 1st kind at $x = c$, if $\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x) \ne f(c)$ then the discontinuity is called removable discontinuity of 1st kind.

In this case if we define a function $g(x)$ such that $g(x) = \begin{cases} f(x), & \text{if } x \ne c \\ \lim\limits_{x\to c} f(x), & \text{if } x = c \end{cases}$, then $g(x)$ will be continuous at $x = c$

Note : A function having a finite number of jumps in a given interval is called a Piece Wise Continuous or Sectionally Continuous function in this interval. For *e.g.* $\{x\}$, $[x]$

(ii) Discontinuity of 2nd kind

If either L.H.L. or R.H.L or both do not exist then discontinuity is said to be of discontinuity of 2nd kind.

THEOREMS ON CONTINUITY

(i) If f & g are two functions which are continuous at $x = c$ then the functions defined by: $F_1(x) = f(x) \pm g(x)$; $F_2(x) = Kf(x)$, K is any real number; $F_3(x) = f(x)\cdot g(x)$ are also continuous at $x = c$. Further, if $g(c)$ is not zero, then $F_4(x) = \frac{f(x)}{g(x)}$ is also continuous at $x = c$.

(ii) If $f(x)$ is continuous & $g(x)$ is discontinuous at $x = a$ then the product function $\phi(x) = f(x)\cdot g(x)$ may or may not be continuous but sum or difference function $\phi(x) = f(x) \pm g(x)$ will necessarily be discontinuous at $x = a$.

(iii) If $f(x)$ and $g(x)$ both are discontinuous at $x = a$ then the product function $\phi(x) = f(x)\cdot g(x)$ is not necessarily be discontinuous at $x = a$ and at most one out of $f(x) + g(x)$ and $f(x) - g(x)$ is continuous at $x = a$.

CONTINUITY OF COMPOSITE FUNCTIONS

If f is continuous at $x = c$ & g is continuous at $x = f(c)$ then the composite $g[f(x)]$ is continuous at $x = c$.

CONTINUITY ON AN INTERVAL

(A) A function f is said to be continuous in (a, b) if f is continuous at each & every point $\in (a, b)$.

(B) A function f is said to be continuous in a closed interval $[a, b]$ if:

(i) f is continuous in the open interval (a, b),

(ii) f is right continuous at 'a'

i.e. $\lim\limits_{x\to a^+} f(x) = f(a)$ = a finite quantity and

(iii) f is left continuous at 'b'

i.e. $\lim\limits_{x\to b^-} f(x) = f(b)$ = a finite quantity.

(C) All Polynomial functions, Trigonometrical functions, Exponential and Logarithmic functions are continuous at every point of their respective domains.

INTERMEDIATE VALUE THEOREM

A function f which is continuous in $[a, b]$ possesses the following properties:

(i) If $f(a)$ & $f(b)$ possess opposite signs, then there exists at least one solution of the equation $f(x) = 0$ in the open interval (a, b).

(ii) If K is any real number between $f(a)$ & $f(b)$, then there exists at least one solution of the equation $f(x) = K$ in the open interval (a, b).

DIFFERENTIABILITY OF A FUNCTION AT A POINT

(i) The right hand derivative of $f(x)$ at $x = a$ denoted by $f'(a^+)$ is defined by:

$$\text{R.H.D.} = f'(a^+) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h},$$

provided the limit exists.

(ii) The left hand derivative of $f(x)$ at $x = a$ denoted by $f'(a^-)$ is defined by:

$$\text{L.H.D.} = f'(a^-) = \lim_{h\to 0^-} \frac{f(a-h)-f(a)}{-h},$$

provided the limit exists.

(iii) A function $f(x)$ is said to be differentiable at $x = a$ if $f'(a^+) = f'(a^-)$ = finite

By definition $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$

RELATION BETWEEN DIFFERENTIABILITY & CONTINUITY

(i) If $f'(a)$ exists then $f(x)$ is continuous at $x = a$.

(ii) If $f(x)$ is differentiable at every point of its domain of definition, then it is continuous in that domain.

Note : The converse of the above result is not true *i.e.* "If 'f' is continuous at $x = a$, then 'f' is differentiable at $x = a$ is need not true.

DIFFERENTIABILTY OF SUM, PRODUCT & COMPOSITION OF FUNCTIONS

(i) If $f(x)$ & $g(x)$ are differentiable at $x = a$ then the functions $f(x) \pm g(x)$, $f(x)\cdot g(x)$ will also be differentiable at $x = a$ & if $g(a) \neq 0$ then the function $f(x)/g(x)$ will also be differentiable at $x = a$.

(ii) If $f(x)$ is not differentiable at $x = a$ & $g(x)$ is differentiable at $x = a$, then the product function $F(x) = f(x)\cdot g(x)$ can still be differentiable at $x = a$.

(iii) If $f(x)$ & $g(x)$ both are not differentiable at $x = a$ then the product function;
$F(x) = f(x)\cdot g(x)$ can still be differentiable at $x = a$.

(iv) If $f(x)$ & $g(x)$ both are non-differentiable at $x = a$ then the sum function $F(x) = f(x) + g(x)$ may be a differentiable function.

DIFFERENTIABILITY ON AN INTERVAL

$f(x)$ is said to be differentiable on an open interval if it is differentiable at each point of the interval and $f(x)$ is said to be differentiable on a closed interval $[a, b]$ if:

(i) for the points a and b, $f'(a^+)$ & $f'(b^-)$ exist finitely

(ii) for any point c such that $a < c < b$, $f'(c+)$ & $f'(c-)$ exist finitely & are equal.

All polynomial, exponential, logarithimic and trigonometric (inverse trigonometric not included) functions are differentiable in their domain.

Note : Derivability should be checked at following points.

(i) At all points where continuity is required to be checked.

(ii) At the critical points of modulus and inverse trigonometric function.

Algebra of Derivatives

(i) $(u \pm v)' = u' \pm v'$

(ii) $(uv)' = u'v + uv'$ (Product rule)

(iii) $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$, where $v \neq 0$ (Quotient rule)

DERIVATIVE OF COMPOSITE FUNCTIONS (CHAIN RULE)

- Let $y = f(t)$ and $t = g(x)$

  Then $\frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right)$
- Let $y = f(t)$, $t = g(x)$ and $u = h(x)$. Then

  $\frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{du} \times \frac{du}{dx}\right)$

This rule may be extended further on more variables and also called chain rule.

DERIVATIVES OF SOME IMPORTANT FUNCTIONS

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
- $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, for $x \in (-1, 1)$
- $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$, for $x \in (-1, 1)$
- $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
- $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$ for $x \in R - [-1, 1]$
- $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$ for $x \in R - [-1, 1]$

DERIVATIVE OF IMPLICIT FUNCTION

When the variables x and y are connected by a relation of the form $f(x, y) = 0$ and it is not possible or convenient to express y as a function of x in the form $y = \phi(x)$, then y is said to be an implicit function of x.

To find $\frac{dy}{dx}$ is such case, we differentiate both sides of given relation with respect to x keeping in mind $\frac{d}{dx}(\phi(y)) = \frac{d\phi}{dy} \times \frac{dy}{dx}$.

For example, $\frac{d}{dx}(\sin y) = \cos y \cdot \frac{dy}{dx}$

DERIVATIVE OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}e^{ax} = ae^{ax}$
- $\frac{d}{dx}(a^x) = a^x \log_e a,\ a > 0$
- $\frac{d}{dx}(\log_e x) = \frac{1}{x}$
- $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$

LOGARITHMIC DIFFERENTIATION

If $y = u^v$ where u and v are functions of x. To find $\frac{dy}{dx}$ for such functions we proceed as follows :

$y = u^v$

Taking log on both sides

$\log y = \log u^v = v \log u$

Differentiating w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = v \frac{1}{u} \cdot \frac{du}{dx} + \log u \cdot \frac{dv}{dx}$$

$$\frac{dy}{dx} = y\left[\frac{v}{u}\frac{du}{dx} + \log u \cdot \frac{dv}{dx}\right]$$

$$\frac{d}{dx}(u^v) = (u^v)\left[\frac{v}{u}\frac{du}{dx} + \log u \cdot \frac{dv}{dx}\right]$$

DERIVATIVE OF PARAMETRIC FUNCTIONS

A relation expressed between two variables x & y in the form $x = f(t)$, $y = g(t)$ is said to be parametric form with t as parameter. The derivative of such function is given by

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{g'(t)}{f'(t)}, \text{ provided } f'(t) \neq 0$$

SECOND ORDER DERIVATIVES

Let $y = f(x)$. Then $\frac{dy}{dx} = f'(x)$

If $f'(x)$ is differentiable, we may differentiate it again w.r.t. x. Then the left hand side becomes $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ which is called the second order derivatives of y w.r.t. x and is denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$.

PROBLEMS

Single Correct Answer Type

1. If $\lim_{x\to 0}\sqrt{\ln\left(\cos\left(x^2+[|a-1|]\right)\right)}=0$ (where [·] is G.I.F.) then range of a contains

(a) [1, 2) (b) (0, 2) (c) [–1, 1] (d) [0, 3)

2. If $f(x)=\cot^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ and $g(x)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

then $\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)},\left(0<a<\frac{1}{2}\right)$ is

(a) $\frac{3}{2\left(1+a^2\right)}$ (b) $\frac{3}{2}$

(c) $\frac{2}{3\left(1+a^2\right)}$ (d) $\frac{-3}{2}$

3. Let $f(x)=\lim_{n\to\infty}\frac{\log_e(2+x)-x^{2n}\sin x}{1+x^{2n}}$ then

(a) $f(x)$ is continuous at $x = 1$

(b) $\lim_{x\to 1^+} f(x)=\log_e 3$

(c) $\lim_{x\to 1^+} f(x)=-\sin 1$

(d) $\lim_{x\to 1^-} f(x)$ does not exist

4. The function $f(x)=\frac{x}{1+|x|}$ is differentiable in

(a) R (b) R – {0}

(c) [0, ∞) (d) (0, ∞)

5. $f : A \to B$ be a function such that $f(x)=\sqrt{x-2}+\sqrt{4-x}$, then set A and B for which $f(x)$ is invertible. Then which of the following is not possible?

(a) A = [2, 4] (b) $B=\left[\sqrt{2},2\right]$

(c) A = [3, 4] (d) A = [2, 3]

6. $f : (-\infty, -1] \to (0, e^5]$ defined by $f(x) = e^{x^3-3x+2}$ is

(a) many one & onto (b) many-one & into

(c) one-one & onto (d) one-one & into

7. $f(x)=\begin{cases}\alpha+\dfrac{\sin[x]}{x} & , \quad x>0\\ 2 & , \quad x=0\\ \beta+\left[\dfrac{\sin x-x}{x^3}\right] & , \quad x<0\end{cases}$

Where [·] is G.I.F. If $f(x)$ is continuous at $x = 0$ then $\beta - \alpha$ equal to

(a) 1 (b) –1 (c) 2 (d) –2

8. Let $f(x) = \cos x$,

$$g(x)=\begin{cases}\text{minimum}\{f(t):0\le t\le x\}, & x\in[0,\pi]\\ \sin x-1, & x>\pi\end{cases},$$

then

(a) $g(x)$ is discontinuous at $x = \pi$

(b) $g(x)$ is continuous at $x = \pi$

(c) $g(x)$ is differentiable at $x = \pi$

(d) $g(x)$ is differentiable for $x \in [0, \infty)$

9. $\underset{x\to\infty}{\text{Lt}}\frac{\sum_{r=1}^{10}(x+r)^{2010}}{(x^{1006}+1)(2x^{1004}+1)}=$

(a) 5 (b) 2010 (c) $\frac{502}{1005}$ (d) 0

10. If graph of the function $y = f(x)$ is continuous and passes through point (3, 1), then $\lim_{x\to 3}\frac{\ln(3f(x)-2)}{2(1-f(x))}$ is equal to

(a) 3/2 (b) 1/2 (c) –3/2 (d) –1/2

11. Let $f(x)=\begin{cases}g(x)\cdot\cos\frac{1}{x}, & \text{if } x\ne 0\\ 0, & \text{if } x=0\end{cases}$ where $g(x)$ is an even function differentiable at $x = 0$, passing through the origin . Then $f'(0)$

(a) is equal to 1 (b) is equal to 0

(c) is equal to 2 (d) does not exist

12. If $f(x)=\frac{x-e^x+\cos 2x}{x^2}$, $x \ne 0$ is continuous at $x = 0$, then

(a) $f(0)=\frac{5}{2}$ (b) $[f(0)] = -2$

(c) $\{f(0)\} = -0.5$ (d) $[f(0)]\cdot\{f(0)\} = -1.5$

where $[x]$ and $\{x\}$ denotes greatest integer and fractional part function respectively.

Multiple Correct Answer Type

13. $f(x)=\left(\frac{x}{2+x}\right)^{2x}$, then

(a) $\lim_{x\to\infty} f(x) = -4$ (b) $\lim_{x\to\infty} f(x) = 2$

(c) $\lim_{x\to\infty} f(x) = e^{-4}$ (d) $\lim_{x\to 1} f(x) = \frac{1}{9}$

14. If $f(x) = \begin{cases} \frac{x}{1+e^{1/x}} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ then

(a) $f'(0^+) = 1$ (b) $f'(0^+) = 0$

(c) $f'(0^-) = 1$ (d) $f'(0^-) = 0$

15. If $f(x) = \begin{cases} \frac{\tan^2\{x\}}{x^2 - [x]^2} & , \text{ if } x > 0 \\ 1 & , \text{ if } x = 0 \\ \sqrt{\{x\}\cot\{x\}} & , \text{ if } x < 0 \end{cases}$ then

where $[x]$ and $\{x\}$ denotes greatest integer and fractional part function respectively.

(a) $\lim_{x\to 0^+} f(x) = 1$ (b) $\lim_{x\to 0^-} f(x) = 1$

(c) $\cot^{-1}\left(\lim_{x\to 0^+} f(x)\right)^2 = 1$

(d) $\tan^{-1}\left(\lim_{x\to 0^+} f(x)\right) = \frac{\pi}{4}$

16. If $f(x) = e^{[\cot x]}$ where $[\cdot]$ denotes the greatest integer function, then

(a) $\lim_{x\to \pi/2^+} f(x) = 1$ (b) $\lim_{x\to \pi/2^+} f(x) = \frac{1}{e}$

(c) $\lim_{x\to \pi/2^-} f(x) = 1$ (d) $\lim_{x\to \pi/2^-} f(x) = \frac{1}{e}$

17. If $f(x) = \sqrt{2-x} + \sqrt{1+x}$, then

(a) $\lim_{x\to a} [f(x)] = 2$ where $a \in [0, 1/2]$ and $[\cdot]$ denotes the greatest integer function.

(b) range of $f(x)$ is $[1+\sqrt{2}, \sqrt{6}]$ when $x \in \left[0, \frac{1}{2}\right]$

(c) both (a) and (b)

(d) none of these

18. Let $a = \min(x^2 + 2x + 3, x \in R)$ and $b = \operatorname{Lim}_{\theta\to 0} \frac{1-\cos\theta}{\theta^2}$.

The value of $\sum_{r=0}^{n} a^r \cdot b^{n-r}$ is

(a) $\frac{2^{n+1}-1}{3\cdot 2^n}$ (b) $\frac{2^{n+1}+1}{3\cdot 2^n}$

(c) $\frac{4^{n+1}-1}{3\cdot 2^n}$ (d) none of these

19. Let $f(x) = \lim_{n\to\infty} \frac{\tan^{-1}(\tan x)}{1+(\log_e x)^n}, x \neq (2n+1)\frac{\pi}{2}$, then

(a) $\forall\, 1 < x < \frac{\pi}{2}, f(x)$ is an identity function.

(b) $\forall\, \frac{\pi}{2} < x < \pi$, the graph of $f(x)$ is a straight line having y intercept of $-\pi$

(c) $\forall\, \frac{\pi}{2} < x < e$, the graph of $f(x)$ is a straight line having y intercept of $-\pi$

(d) $\forall\, x > e, f(x)$ is a constant function

20. If $f(x) = \lim_{n\to\infty} (2\sin x)^n$ for all $x \in \left[0, \frac{\pi}{6}\right]$. Then which of the following statements is true?

(a) $f\left(\frac{\pi}{6}\right)$ is not equal to 1

(b) $f(x)$ has irremovable discontinuity at $x = \frac{\pi}{6}$

(c) $f(x)$ has removable discontinuity at $x = \frac{\pi}{6}$

(d) $f(x)$ is continuous in $\left(0, \frac{\pi}{6}\right)$

Comprehension Type

Paragraph for Q. No. 21 to 23

$f(x) = x^2 + xg'(1) + g''(2)$ and $g(x) = f(1)x^2 + xf'(x) + f''(x)$

21. The value of $f(3)$ is

(a) 1 (b) 0 (c) -1 (d) -2

22. The value of $g(0)$ is

(a) 0 (b) -3 (c) 2 (d) 1

23. The domain of function $\sqrt{\frac{f(x)}{g(x)}}$ is

(a) $(-\infty, 1] \cup (2, 3]$ (b) $(-2, 0] \cup (1, \infty)$

(c) $(-\infty, 0] \cup \left(\frac{2}{3}, 3\right]$ (d) $(-\infty, \infty)$

Paragraph for Q. No. 24 to 26

Let $f(x)$ be a differentiable function, we define a mathematical operation

$G^*(f(x)) = \operatorname{Lim}_{h\to 0} \frac{f^m(x+h) - f^m(x)}{h}$, where m is a fixed natural number.

24. $G^*(f(x)+g(x))$ is equal to

(a) $$G^*(f(x))\left[\frac{(f(x)+g(x))^{m-1}}{(f(x))^{m-1}}\right]+G^*(g(x))\left[\frac{(f(x)+g(x))^{m-1}}{(g(x))^{m-1}}\right]$$

(b) $$G^*(f(x))\left[\frac{(f(x)+g(x))^{m-1}}{(f(x))^{m-1}}\right]-G^*(g(x))\left[\frac{(f(x)+g(x))^{m-1}}{(g(x))^{m-1}}\right]$$

(c) $$G^*(f(x))\left[\frac{(f(x)-g(x))^{m-1}}{(f(x))^{m-1}}\right]+G^*(g(x))\left[\frac{(f(x)-g(x))^{m-1}}{(g(x))^{m-1}}\right]$$

(d) none of these

25. $G^*(f(x)\cdot g(x))$ is equal to

(a) $(G^*(f(x)))\cdot g(x)^m-(G^*(g(x)))\cdot f(x)^m$

(b) $(G^*(f(x)))\cdot g(x)^m+(G^*(g(x)))\cdot f(x)^m$

(c) $(G^*(f(x)))\cdot g(x)^{m-1}+(G^*(g(x)))\cdot f(x)^{m-1}$

(d) none of these

26. $G^*f(g(x))$ is equal to

(a) $m\,f'(g(x)).g'(x)\;\; f^{m-1}(g(x))$

(b) $m\,f'(g(x)).g'(x)\;\; f^{m}(g(x))$

(c) $m\,f'(g(x)).g(x)\;\; f^{m-1}(g(x))$

(d) none of these

Paragraph for Q. No. 27 to 29

Suppose f, g, h are real-valued functions defined on some deleted neighbourhood N of 'a' such that $g(x)\le f(x)\le h(x)$ for all $x\in N$.

If $\lim\limits_{x\to a} g(x)$ and $\lim\limits_{x\to a} h(x)$ both exist and each equals λ, then $\lim\limits_{x\to a} f(x)$ also exists and equals λ. (This is called the squeezing principle or sandwich theorem)

27. If $f(x)=\sin(\sqrt{x+1})-\sin(\sqrt{x})$, then $\lim\limits_{x\to\infty} f(x)$

(a) does not exist (b) equals 0
(c) equals 1 (d) equals 1/2

28. If $f(x)=[(\tan x+\sin x)^2]$, where $[\cdot]$ is the greatest integer function, then $\lim\limits_{x\to 0} f(x)=$

(a) 0 (b) 1 (c) 2 (d) 1/2

29. $\lim\limits_{x\to 0}\{1^{\operatorname{cosec}^2 x}+2^{\operatorname{cosec}^2 x}+\ldots+n^{\operatorname{cosec}^2 x}\}^{\sin^2 x}$

(a) equals 0 (b) equals 1
(c) equals n (d) does not exist

Matrix-Match Type

30. Match the following :

	Column-I		Column-II
(A)	$\lim\limits_{x\to 0^+}(\operatorname{cosec} x)^{1/\ln x}=$	(p)	1
(B)	$\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}(\sec x)^{\cot x}=$	(q)	$e^{1/3}$
(C)	$\lim\limits_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}=$	(r)	$1/e$
(D)	$\lim\limits_{x\to 0}(\cos x)^{1/x^2}=$	(s)	$\sqrt{e}$

Integer Answer Type

31. If $k=\lim\limits_{x\to 0}\dfrac{x\tan 2x-2x\tan x}{(1-\cos 2x)^2}$; then $2k=$

32. If $\tan\cos^{-1}x=\sin\cot^{-1}\dfrac{1}{2}$ has a solution $x=\dfrac{\sqrt{5}}{k}$. The value of k is

33. If $\lim\limits_{x\to 0}\dfrac{\sin x^4-x^4\cos x^4+x^{20}}{x^4(e^{2x^4}-1-2x^4)}=k$, then $6k=$

34. Let $f:R\to R$ be the function defined by $f(x)=\max\{x,x^3\}$. The number of points in R where f is not differentiable is

35. Let $f(x)$ be a function such that $\lim\limits_{x\to 0}\dfrac{f(x)}{x}=1$. If $\lim\limits_{x\to 0}\dfrac{x(1+a\cos x)-b\sin x}{\{f(x)\}^3}=1$, then $|(a+b)|=$

36. $\underset{x\to 0}{\text{Lt}}\dfrac{3\log\left(\frac{3+x^2}{3-x^2}\right)}{x^2}$ is equal to

37. If $f(x+y)=f(x)f(y)$ and $f(x)=1+xg(x)H(x)$ where $\underset{x\to 0}{\text{Lt}}\,g(x)=2$, $\underset{x\to 0}{\text{Lt}}\,H(x)=3$, then $f'(x)=Kf(x)$. Find the value of K.

38. $\underset{x\to 0}{\text{Lt}} \dfrac{1-\cos^n(1-\cos x)}{\tan^m x}=1$. Then $\dfrac{n}{m}=$

39. If $f(x)$ is a continuous function $\forall\ x\in R$ and the range of $f(x)$ is $(2,\sqrt{21})$ and $g(x)=\left[\dfrac{f(x)}{c}\right]$ is continuous $\forall\ x\in R$, then the least positive integral value of c is (here $[x]$ is greatest integer $\le x$)

40. Let $f(x)=\begin{cases}-1, & -2\le x\le 0\\ x-1, & 0<x\le 2\end{cases}$. Then the number of points where the function $g(x) = f(|x|) + |f(x)|$ is not differentiable is

SOLUTIONS

1. **(b) :** $\lim\limits_{x\to 0}\sqrt{\ln\cos\left(x^2+[|a-1|]\right)}=0$

$=\lim\limits_{x\to 0}\left([|a-1|]\right)=0;\ 0\le|a-1|<1.$

$\Rightarrow\ 0 < a < 2$

2. **(d) :** $\lim\limits_{x\to a}\dfrac{-3\left(\tan^{-1}x-\tan^{-1}a\right)}{2\left(\tan^{-1}x-\tan^{-1}a\right)}=-\dfrac{3}{2}$

3. **(c) :** For $|x| < 1$, $x^{2n} \to 0$ as $n \to \infty$

$|x|>1, \dfrac{1}{x^{2n}}\to 0$ as $n\to\infty$

$$f(x)=\begin{cases}\log_e(2+x), & |x|<1\\ \lim\limits_{n\to\infty}\dfrac{x^{-2n}\log_e(2+x)-\sin x}{x^{-2n}+1}=-\sin x, & \text{if } |x|>1\\ \dfrac{1}{2}\left(\log_e(2+x)-\sin x\right), & |x|=1\end{cases}$$

$\lim\limits_{x\to 1^+} f(x)=-\sin 1;\ \lim\limits_{x\to 1^-} f(x)=\log 3.$

4. **(a) :** The function $f(x)$ is an odd function with Range $(-1, 1) \Rightarrow$ it is differentiable everywhere.

5. **(a) :** Domain is $[2, 4]$ and range is $\left[\sqrt{2},2\right]$. $f'(x)=0\Rightarrow x=3\Rightarrow f(3)=2$. Domain has to be restricted to $[2, 3]$ or $[3, 4]$ for the function to be invertible.

6. **(d) :** $f(x) = e^{x^3-3x+2}$

$f'(x)=e^{x^3-3x+2}\left(3x^2-3\right)\ge 0\,\forall x\in(-\infty,-1]$

$f(x)$ has its domain $(-\infty, -1]$, in its domain $f(x)$ is one-one and range is $(0, e^4]$.

7. **(a) :** R.H.L. $(x = 0) = \alpha + 0 = \alpha$

Now, $\dfrac{\sin x-x}{x^3}=\dfrac{x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-......-x}{x^3}=\dfrac{-1}{3!}+\dfrac{x^2}{5!}-.....$

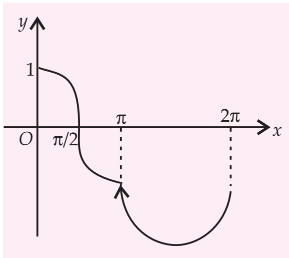
$\lim\limits_{x\to 0}\dfrac{\sin x-x}{x^3}=\dfrac{-1}{6}$

L.H.L. $= \beta - 1$

$f(x)$ is continuous at $x = 0 \Rightarrow \beta - 1 = 2 = \alpha$

$\Rightarrow\ \beta = 3,\ \alpha = 2$. So, $\beta - \alpha = 1$

8. **(b) :** $g(x)=\begin{cases}\cos x, & 0\le x\le\pi\\ \sin x-1, & x>\pi\end{cases}$



Given figure represents the graph of $g(x)$. Clearly, $g(x)$ is continuous but non-differentiable at $x = \pi$.

9. **(a) :** Degree of numerator = degree of denominator = 2010.

$\therefore$ Required limit $=\lim\limits_{x\to\infty}\dfrac{10x^{2010}+....}{2x^{2010}+....}=5$

10. **(c) :** Since $\lim\limits_{x\to 3} f(x)=1$

$\therefore\ \lim\limits_{x\to 3}\dfrac{\ln(1+3(f(x)-1))}{-2(f(x)-1)}=-\dfrac{3}{2}$

11. **(b) :** $f'(0)=\underset{h\to 0}{\text{Lim}}\dfrac{g(0+h)\cos(1/h)-0}{h}$

$=\underset{h\to 0}{\text{Lim}}\dfrac{g(h)\cos(1/h)}{h}=\underset{h\to 0}{\text{Lim}}\ g'(0)\cos(1/h)=0$

$\because\ g'(x) = -g'(x)\ \Rightarrow\ g'(0) = 0$

12. **(d) :** $\underset{x\to 0}{\text{Lim}}\dfrac{x-e^x+1-(1-\cos 2x)}{x^2}=-\dfrac{1}{2}-2=-\dfrac{5}{2}$

hence for continuity $f(0)=-\dfrac{5}{2}$

$\therefore\ [f(0)]=\left[-\dfrac{5}{2}\right]=-3;\ \{f(0)\}=\left\{-\dfrac{5}{2}\right\}=\dfrac{1}{2}$

hence $[f(0)].\{f(0)\}=-\dfrac{3}{2}=-1.5$

13. (c, d) : $f(x)=\left(\frac{x}{2+x}\right)^{2x}$

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\left(\frac{x}{2+x}\right)^{2x}=\lim_{x\to\infty}\left(1+\frac{x}{2+x}-1\right)^{2x}$$

$$e^{\lim_{x\to\infty} 2x\left(-\frac{2}{2+x}\right)}=e^{\lim_{x\to\infty} -4\left(\frac{x}{2+x}\right)}=e^{-4}$$

Also $\lim_{x\to 1} f(x)=\left(\frac{1}{3}\right)^2=\frac{1}{9}$.

14. (b, c) : $f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x}=\lim_{x\to 0}\frac{1}{1+e^{1/x}}$

$$f'(0^+)=\lim_{x\to 0^+}\frac{1}{1+e^{1/x}}=0$$

$$f'(0^-)=\lim_{x\to 0^-}\frac{1}{1+e^{1/x}}=1.$$

15. (a, d) : $x\to 0^+, \{x\}=x; [x]=0; f(x)=\frac{\tan^2 x}{x^2}\to 1$

$\therefore$ (a), (d) are true

$x\to 0^-; \{x\}=x+1; [x]=-1;$

$f(x)=\sqrt{(x+1)\cot(x+1)}\to 1.$

16. (b, c) :

$$\lim_{x\to\frac{\pi}{2}^-} f(x)=e^{\lim_{h\to 0}\left[\cot\left(\frac{\pi}{2}-h\right)\right]}=e^{\lim_{h\to 0}[\tan h]}=1$$

$$\lim_{x\to\frac{\pi}{2}^+} f(x)=e^{\lim_{h\to 0}\left[\cot\left(\frac{\pi}{2}+h\right)\right]}=e^{\lim_{h\to 0}[-\tan h]}=e^{-1}=\frac{1}{e}$$

17. (c) :

(a) $f(x)=\sqrt{2-x}+\sqrt{1+x},\ x\in[-1, 2]$

$\lim_{x\to a}[f(x)]=\left[\sqrt{2-a}+\sqrt{1+a}\right]=2$, as $a\in\left[0, \frac{1}{2}\right]$.

(b) $f(0)=1+\sqrt{2}, f\left(\frac{1}{2}\right)=\sqrt{6}$

Range of $f(x)=\left[1+\sqrt{2}, \sqrt{6}\right]$.

18. (c) : $a = \min(x^2 + 2x + 3, x \in R)$

i.e., $x^2 + 2x + 3 = x^2 + 2x + 1 + 2 = (x + 1)^2 + 2$

$\therefore$ a is minimum when $x = -1$, *i.e.*, $a = 2$.

Again, $b=\lim_{\theta\to 0}\frac{2\sin^2\theta/2}{\theta^2}=\lim_{\theta\to 0}\frac{2}{4}\cdot\frac{\sin^2\theta/2}{\theta^2/4}$

$$=\lim_{\theta\to 0}\frac{1}{2}\cdot\frac{\sin^2\theta/2}{(\theta/2)^2}=\frac{1}{2}$$

Hence, $\sum_{r=0}^{n} a^r\cdot b^{n-r}=\sum_{r=0}^{n} 2^r\cdot\left(\frac{1}{2}\right)^{n-r}$,

$$=\frac{1}{2^n}(2^0+2^2+2^4+\ldots+2^{2n})=\frac{4^{n+1}-1}{3\cdot 2^n}.$$

19. (a, c, d) : $\forall\ 1<x<\frac{\pi}{2}$, $\tan^{-1}(\tan x) = x$ &

$0<\log_e x<\log_e\frac{\pi}{2}<1$

$\Rightarrow$ $f(x) = x$ $\quad$ $[\because (\log_e x)^n\to 0]$

$\forall\ \frac{\pi}{2}<x<e,\ \tan^{-1}(\tan x)=x-\pi$

& $0 < \log_e x < 1$ $\quad\therefore$ $(\log_e x)^n\to 0$

$\Rightarrow$ $f(x) = x - \pi$

and $x > e$, $\log_e x > 1$ $(\log_e x)^n\to\infty$

$\Rightarrow$ $f(x) = 0$

20. (c, d) : Since $0 \le 2\sin x < 1$ for $0\le x<\frac{\pi}{6}$,

$$f(x)=\begin{cases}0, & 0\le x<\frac{\pi}{6}\\ 1, & x=\frac{\pi}{6}\end{cases}$$

Clearly $f(x)$ has removable discontinuity at $x=\frac{\pi}{6}$ and $f(x)$ is continuous in $\left(0, \frac{\pi}{6}\right)$.

21. (b): Here put $g'(1) = a$, $g''(2) = b$ ...(i)

Then $f(x) = x^2 + ax + b$, $f(1) = 1 + a + b$

$\Rightarrow$ $f'(x) = 2x + a$, $f''(x) = 2$

$\therefore$ $g(x) = (1 + a + b)x^2 + (2x + a)x + 2$

$= x^2(3 + a + b) + ax + 2$

$\Rightarrow$ $g'(x) = 2x(3 + a + b) + a$ and $g''(x) = 2(3 + a + b)$

Hence, $g'(1) = 2(3 + a + b) + a$ ...(ii)

$g''(2) = 2(3 + a + b)$ ...(iii)

Form (i), (ii) and (iii), we have

$a = 2(3 + a + b) + a$ and $b = 2(3 + a + b)$

$\Rightarrow$ $3 + a + b = 0$ and $b + 2a + 6 = 0$

Hence $b = 0$ and $a = -3$. So, $f(x) = x^2 - 3x \Rightarrow f(3) = 0$

22. (c) : $g(x) = -3x + 2 \Rightarrow g(0) = 2$

23. (c) : $\sqrt{\dfrac{f(x)}{g(x)}} = \sqrt{\dfrac{x^2-3x}{-3x+2}}$ is defined if $\dfrac{x^2-3x}{-3x+2} \geq 0$

$$\Rightarrow \frac{x(x-3)}{\left(x-\frac{2}{3}\right)} \leq 0 \Rightarrow x \in (-\infty, 0] \cup \left(\frac{2}{3}, 3\right]$$

24. (a): $G^*(f(x)) = \lim\limits_{h\to 0} \dfrac{f^m(x+h)-f^m(x)}{h} = \dfrac{d}{dx}(f(x))^m$

$$= mf^{m-1}(x).f'(x)$$

$$G^*(f(x)+g(x)) = m(f(x)+g(x))^{m-1}.(f'(x)+g'(x))$$

$$= m(f(x)+g(x))^{m-1}.f'(x) + m(f(x)+g(x))^{m-1}.g'(x)$$

$$= m(f(x)+g(x))^{m-1}.\frac{G^*(f(x))}{mf^{m-1}(x)} + m(f(x)$$

$$+g(x))^{m-1}.\frac{G^*(g(x))}{mg^{m-1}(x)}$$

$$= G^*(f(x))\left[\frac{(f(x)+g(x))^{m-1}}{f^{m-1}(x)}\right]$$

$$+G^*(g(x))\left[\frac{(f(x)+g(x))^{m-1}}{(g(x))^{m-1}}\right]$$

25. (b) **26. (a)**

27. (b): $x > 0 \Rightarrow 0 \leq \left|\sin\sqrt{x+1} - \sin\sqrt{x}\right|$

$$= \left|2\cos\left(\frac{\sqrt{x+1}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+1}-\sqrt{x}}{2}\right)\right|$$

$$\leq 2\left|\sin\frac{1}{2(\sqrt{x+1}+\sqrt{x})}\right|$$

Since, $\dfrac{1}{2(\sqrt{x+1}+\sqrt{x})} \to 0$ as $x \to +\infty$ the given limit equals 0.

28. (a) $0 \leq (\tan x + \sin x)^2$

$$= (\tan^2 x)(1+\cos x)^2 < 1 \text{ for } \frac{-\pi}{8} < x < \frac{\pi}{8}$$

$$\therefore\ f(x) = 0 \text{ in } \left[\frac{-\pi}{8}, \frac{\pi}{8}\right].$$

29. (c) $f(x) = \left(\sum\limits_{r=1}^{n} r^{\operatorname{cosec}^2 x}\right)^{\sin^2 x} \geq n$

for $\dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$ and $x \neq 0$

$\Rightarrow f(x) \leq \left(n \cdot n^{\operatorname{cosec}^2 x}\right)^{\sin^2 x} = n^{1+\sin^2 x}$ for $\dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$ and $x \neq 0$

$\therefore\ n \leq f(x) \leq n^{1+\sin^2 x}$ for $x \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$ and $x \neq 0$

$\therefore \lim\limits_{x\to 0} f(x) = n$

30. A → r; B → p; C → q; D → s

(A) $y = (\sin x)^{1/\ln x}$, $\ln y = \dfrac{\ln \sin x}{\ln x}$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\cot x}{1/x} = \lim_{x\to 0^+} \frac{x}{\tan x} = 1$$

$y \to e$ ∴ Required limit = $1/e$.

(B) $y = (\cos x)^{\cot x}$; $\ln y = \cot x . \ln \cos x$

$$\lim_{x\to\frac{\pi}{2}^-} \ln y = \lim_{x\to\frac{\pi}{2}^-} \frac{\ln \cos x}{\tan x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\tan x}{\sec^2 x} = \lim_{x\to\frac{\pi}{2}^-} \sin x \cos x = 0;$$

$y \to 1$ ∴ Required limit = 1.

(C) $y = \left(\dfrac{\tan x}{x}\right)^{1/x^2}$; $\ln y = \dfrac{\ln \tan x - \ln x}{x^2}$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \left(\frac{\sec^2 x}{\tan x} - \frac{1}{x}\right).\frac{1}{2x}$$

$$= \lim_{x\to 0}\left(\frac{1}{\sin 2x} - \frac{1}{2x}\right).\frac{1}{x} = \lim_{x\to 0} .\frac{2x - \sin 2x}{2x.\sin 2x.x}; \text{ (Put } 2x = \theta)$$

$$= 2\lim_{\theta\to 0} \frac{\theta - \sin\theta}{\theta^2 \sin\theta} = 2\lim_{\theta\to 0} \frac{\theta - \sin\theta}{\theta^3}.\frac{\theta}{\sin\theta} = \frac{1}{3}$$

$$\Rightarrow y \to e^{1/3}$$

(D) $y = (\cos x)^{1/x^2}$; $\ln y = \dfrac{1}{x^2} \ln \cos x$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln \cos x}{x^2} = \lim_{x\to 0} \frac{\tan x}{2x} = \frac{1}{2},\ y \to \sqrt{e}$$

31. (1) : $k = \lim\limits_{x\to 0}\left(x \cdot \dfrac{2\tan x}{1-\tan^2 x} - 2x\tan x\right).\dfrac{1}{4\sin^4 x}$

$$= \frac{1}{4}.\lim_{x\to 0} \frac{2x\tan x}{\sin^4 x}\left(\frac{\tan^2 x}{1-\tan^2 x}\right)$$

$$=\frac{1}{2}\lim_{x\to 0}\frac{x}{\sin x}\cdot\frac{1}{\cos^3 x}\cdot\frac{1}{1-\tan^2 x}=\frac{1}{2}$$

$\therefore\quad 2k = 1.$

32. (3) : $\tan\cos^{-1}x=\sin\sin^{-1}\frac{2}{\sqrt{5}}=\frac{2}{\sqrt{5}}>0$

$\cos^{-1}x\in\left(0,\frac{\pi}{2}\right);x\in(0,1)$

$\frac{\sqrt{1-x^2}}{x}=\frac{2}{\sqrt{5}},\ \frac{1-x^2}{x^2}=\frac{4}{5};x^2=\frac{5}{9},\ x=\frac{\sqrt{5}}{3};k=3.$

33. (1): $\theta = x^4 \to 0$

$$k=\lim_{\theta\to 0}\frac{\sin\theta-\theta\cos\theta+\theta^5}{\theta(e^{2\theta}-1-2\theta)}$$

$$k=\lim_{\theta\to 0}\left(\frac{\sin\theta-\theta}{\theta^3}+\frac{\theta(1-\cos\theta)}{\theta^3}+\frac{\theta^5}{\theta^3}\right)\cdot\left(\frac{\theta^2}{e^{2\theta}-1-2\theta}\right)$$

Now, $\lim_{\theta\to 0}\frac{\sin\theta-\theta}{\theta^3}=\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}\left(-\frac{1}{3}\right)=-\frac{1}{6}$

and $\lim_{\theta\to 0}\left(\frac{1-\cos\theta}{\theta^2}\right)=\lim_{\theta\to 0}\frac{\sin\theta}{2\theta}=\frac{1}{2}$

$$\lim_{\theta\to 0}\frac{e^{2\theta}-1-2\theta}{\theta^2}=\lim_{\theta\to 0}\frac{2e^{2\theta}-2}{2\theta}=\lim_{\theta\to 0}2e^{2\theta}=2$$

So, $k=\left(-\frac{1}{6}+\frac{1}{2}+0\right)\left(\frac{1}{2}\right)=\frac{1}{6}$

$\Rightarrow\quad 6k = 1.$

34. (3) : $f(x)=\begin{cases}x, & \text{if } x\le -1\\ x^3, & \text{if } -1\le x\le 0\\ x, & \text{if } 0\le x\le 1\\ x^3, & \text{if } x>1\end{cases}$

Clearly f is not differentiable at $x = -1$, 0 and 1.

35. (4): Given, $\lim_{x\to 0}\frac{f(x)}{x}=1$ ...(1)

$$\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{(f(x))^3}=1$$

$$\Rightarrow\lim_{x\to\infty}\frac{\left[x\left\{1+a\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\ldots\ldots\right)\right\}-b\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\ldots\right)\right]}{\left\{\frac{f(x)}{x}\right\}^3\cdot x^3}=1$$

$$\Rightarrow\lim_{x\to 0}\frac{(1+a-b)x+\left(\frac{b}{3!}-\frac{a}{2!}\right)x^3+\left(\frac{a}{4!}-\frac{b}{5!}\right)x^5+\ldots}{x^3}=1$$

[From (1)]

This is possible only when lowest power of x in the numerator is 3.

$\therefore\quad 1 + a - b = 0\quad$ or $\quad a - b = -1$ ...(2)

Also, limit is 1 $\therefore\ \frac{b}{3!}-\frac{a}{2!}=1\ \Rightarrow\ b-3a=6$ ... (3)

Solving (2) and (3), we get $a=-\frac{5}{2}$ and $b=-\frac{3}{2}$

$\therefore\quad a + b = -4$

36. (2): $\mathop{\mathrm{Lt}}_{x\to 0}\frac{3\{\log(3+x^2)-\log(3-x^2)\}}{x^2}$

$$=\mathop{\mathrm{Lt}}_{x\to 0}\frac{3\left\{\frac{2x}{3+x^2}+\frac{2x}{3-x^2}\right\}}{2x}$$

$$=\mathop{\mathrm{Lt}}_{x\to 0}\frac{18}{9-x^4}=2$$

37. (6): $f'(x)=\mathop{\mathrm{Lt}}_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\mathop{\mathrm{Lt}}_{h\to 0}\frac{f(x)f(h)-f(x)}{h}=\mathop{\mathrm{Lt}}_{h\to 0}\frac{f(x)[1+h\,g(h)H(h)-1]}{h}$$

$$=f(x)\mathop{\mathrm{Lt}}_{h\to 0}g(h)\cdot H(h)=6f(x)$$

38. (2):

$$\mathop{\mathrm{Lt}}_{x\to 0}\frac{1-\cos^n(1-\cos x)}{(1-\cos x)^2}\cdot\left(\frac{1-\cos x}{x^2}\right)^2\cdot\frac{x^4}{\tan^m x}=1$$

$\Rightarrow\quad m = 4$ and $\frac{n}{2}\cdot\frac{1}{4}=1\Rightarrow n=8$

39. (5): $g(x)$ must assume only one integral value *i.e.*

$g(x) = 0\ \forall\ x\in R$

$$\frac{2}{c}<\frac{f(x)}{c}<\frac{\sqrt{21}}{c}$$

$$\frac{\sqrt{21}}{c}<1$$

$c = 5, 6, 7$ ... least value = 5

40. (2)

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CLASS XII | Series 5

ACE YOUR WAY CBSE

Integrals | Application of Integrals

HIGHLIGHTS

INTEGRALS

Let $f(x)$ be a function such that $\frac{d}{dx}(F(x)+C)=f(x)$, then $\int f(x)dx=F(x)+C$

is called the indefinite integral of $f(x)$ w.r.t. x.

SOME FUNDAMENTAL INTEGRATION FORMULAS

(i) $\int x^n dx=\frac{x^{n+1}}{n+1}+C, n\neq -1$

(ii) $\int \frac{1}{x}dx=\log|x|+C$

(iii) $\int e^x dx=e^x+C$

(iv) $\int a^x dx=\frac{a^x}{\log a}+C$

(v) $\int \sin x\ dx=-\cos x+C$

(vi) $\int \cos x\ dx=\sin x+C$

(vii) $\int \sec^2 x\ dx=\tan x+C$

(viii) $\int \operatorname{cosec}^2 x\ dx=-\cot x+C$

(ix) $\int \sec x\tan x\ dx=\sec x+C$

(x) $\int \operatorname{cosec} x\cot x\ dx=-\operatorname{cosec} x+C$

(xi) $\int \frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\left(\frac{x}{a}\right)+C$

Previous Years Analysis						
	2016		2015		2014	
	Delhi	AI	Delhi	AI	Delhi	AI
VSA	–	–	–	–	1	2
SA	3	3	3	3	1	1
LA	1	1	1	1	2	2

(xii) $\int -\frac{1}{\sqrt{a^2-x^2}}dx=\cos^{-1}\left(\frac{x}{a}\right)+C$

(xiii) $\int \frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

(xiv) $\int -\frac{1}{a^2+x^2}dx=\frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right)+C$

(xv) $\int \frac{1}{x\sqrt{x^2-a^2}}dx=\frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+C$

(xvi) $\int -\frac{1}{x\sqrt{x^2-a^2}}dx=\frac{1}{a}\operatorname{cosec}^{-1}\left(\frac{x}{a}\right)+C$

ALGEBRA OF INTEGRATION

For any two functions $f(x)$ and $g(x)$, we have

- $\int(f(x)\pm g(x))dx=\int f(x)dx\pm\int g(x)dx$
- For any constant k, $\int k\cdot f(x)dx=k\int f(x)dx$

METHODS OF INTEGRATION

There are the following three methods for finding integral of a function :

- Integration by substitution
- Integration by partial fractions
- Integration by parts

INTEGRATION BY SUBSTITUTION

Let $I = \int f(x)dx$, if we substitute $x = \phi(t)$, then $dx = \phi'(t)dt$ and the integral is transformed to $I = \int f[\phi(t)]\cdot\phi'(t)dt$

Some Important Substitutions

	Integrals	Substitutions
(i)	$\int f(ax+b)dx$	Put $ax + b = t$
(ii)	$\int x^{n-1} f(x^n)dx$	Put $x^n = t$
(iii)	$\int \{f(x)\}^n f'(x)dx$ or $\int \frac{f'(x)}{f(x)}dx$	Put $f(x) = t$
(iv)	$\int \frac{dx}{ax^2+bx+c}$ or $\int \frac{dx}{\sqrt{ax^2+bx+c}}$	Transformed into standard form by expressing $ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]$ then put $x+\frac{b}{2a}=t$
(v)	$\int (ax+b)\sqrt{cx+d}\,dx$ or $\int \frac{ax+b}{\sqrt{cx+d}}dx$	Put $ax + b = A(cx + d) + B$
(vi)	$\int \frac{dx+e}{ax^2+bx+c}dx$ or $\int \frac{dx+e}{\sqrt{ax^2+bx+c}}dx$ or $\int (dx+e)\sqrt{ax^2+bx+c}\,dx$	Put $(dx+e) = A\frac{d}{dx}(ax^2+bx+c)+B$
(vii)	$\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}dx$	Put $a\sin x + b\cos x = A\frac{d}{dx}(c\sin x + d\cos x) + B(c\sin x + d\cos x)$

INTEGRATION BY PARTIAL FRACTIONS

If $f(x)$ and $g(x)$ are two polynomials and $g(x) \neq 0$, then $\frac{f(x)}{g(x)}$ defines a rational algebraic function or a rational function of x.

(i) If degree of $f(x)$ < degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called a proper rational function.

(ii) If degree of $f(x) \geq$ degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called an improper rational function and express the function $\frac{f(x)}{g(x)}$ as Quotient + $\frac{\text{Remainder}}{g(x)}$, $g(x) \neq 0$

Then, $\frac{\text{Remainder}}{g(x)}$ is a proper rational function.

Rational form	Partial form
$\frac{px+q}{(x-a)(x-b)}$	$\frac{A}{(x-a)}+\frac{B}{(x-b)}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)}+\frac{B}{(x-b)}+\frac{C}{(x-c)}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{(x-a)}+\frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)}+\frac{B}{(x-a)^2}+\frac{C}{(x-b)}$
$\frac{px^2+qx+r}{(x-a)^3(x-b)}$	$\frac{A}{(x-a)}+\frac{B}{(x-a)^2}+\frac{C}{(x-a)^3}+\frac{D}{(x-b)}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{(x-a)}+\frac{Bx+C}{x^2+bx+c}$ where x^2+bx+c cannot be factorised further

INTEGRATION BY PARTS

- When the integrand can be expressed as a product of two functions, one of which can be differentiated and the other can be integrated, then we apply integration by parts.
- If $f(x)$ = first function (that can be differentiated) and $g(x)$ = second function (that can be integrated), then the preference of this order can be decided by the word "*ILATE*", where
  I stands for Inverse Trigonometric Function
  L stands for Logarithmic Function
  A stands for Algebraic Function
  T stands for Trigonometric Function
  E stands for Exponential Function, then

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int\left\{\frac{d}{dx}f(x)\int g(x)dx\right\}dx$$

Some Particular Integrals

(i) $\int \cot x\,dx = \log|\sin x| + C = -\log \operatorname{cosec} x + C$

(ii) $\int \tan x\,dx = -\log|\cos x| + C = \log|\sec x| + C$

(iii) $\int \sec x\,dx = \log|\sec x + \tan x| + C$

(iv) $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C$

(v) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

(vi) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

(vii) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C$

(viii) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + C$

(ix) $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left|x+\sqrt{a^2+x^2}\right| + C$

(x) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$

(xi) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

Special Integrals

(i) $\int e^x[f(x) + f'(x)]dx = e^x f(x) + C$

(ii) $\int e^{kx}[kf(x) + f'(x)]dx = e^{kx} f(x) + C$

DEFINITE INTEGRAL

Let $F(x)$ be the primitive or antiderivative of a continuous function $f(x)$ defined on $[a, b]$ *i.e.*, $\frac{d}{dx}\{F(x)\} = f(x)$. Then the definite integral of $f(x)$ over $[a, b]$ is denoted by $\int_a^b f(x)dx$, where 'a' is called the lower limit and 'b' is called the upper limit. The interval $[a, b]$ is called the interval of integration.

DEFINITE INTEGRAL AS THE LIMIT OF A SUM

Let f be a continuous function defined in a closed interval $[a, b]$, then the definite integral $\int_a^b f(x)dx$ is the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis.

Also,

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + \ldots + f\{a+(n-1)h\}]$$

where, $h = \frac{b-a}{n}, h \to 0$ as $n \to \infty$

FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

First fundamental theorem	Let $f(x)$ be a continuous function on the closed interval $[a, b]$ and let $A(x)$ be the area function. Then $A'(x) = f(x)$, for all $x \in [a, b]$.
Second fundamental theorem	Let $f(x)$ be a continuous function defined on $[a, b]$ and $F(x)$ be the integral of $f(x)$, then $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

PROPERTIES OF DEFINITE INTEGRALS

(i) $\int_a^b f(x)dx = \int_a^b f(y)dy$

(ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$ **(iii)** $\int_a^a f(x)dx = 0$

(iv) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, a < c < b$

(v) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

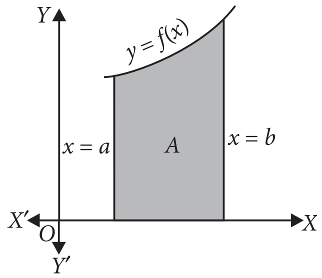
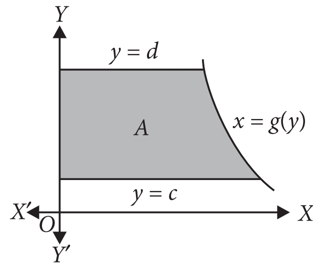
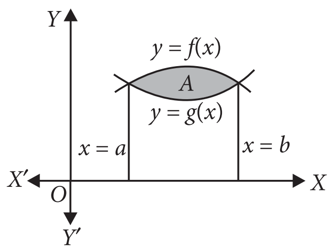
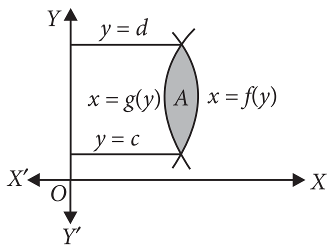
(vi) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(vii) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

(viii) $\int_{-a}^{a} f(x)dx = \begin{cases} 0, & \text{if } f(x) \text{ is an odd function} \\ 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is an even function} \end{cases}$

(ix) $\int_0^{2a} f(x)dx = \begin{cases} 0, & \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \end{cases}$

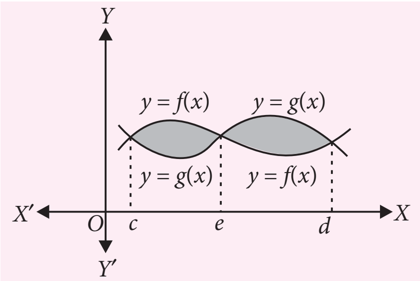
APPLICATION OF INTEGRALS

	Explanation	Geometrical Representation
Area under Simple Curves	Let $y = f(x)$ be a continuous function defined on $[a, b]$. Area bounded by the curve $y = f(x)$, X-axis and bounded between the lines $x = a$ and $x = b$ is given by $A = \int_a^b y dx = \int_a^b f(x) dx$	
	Let $x = g(y)$ be a continuous function defined on $[a, b]$. Area bounded by the curve $x = g(y)$, Y-axis and bounded between the lines $y = c$ and $y = d$ is given by $A = \int_c^d x dy = \int_c^d g(y) dy$	
Area between Two Curves	The area bounded by the two curves $y = f(x)$, $y = g(x)$ is obtained by $A = \left\lvert \int_a^b f(x)dx - \int_a^b g(x)dx \right\rvert$ where $(a, f(a))$ and $(b, f(b))$ are intersecting points of $y = f(x)$ and $y = g(x)$.	
	The area bounded by the two curves $x = f(y)$, $x = g(y)$ is obtained by $A = \left\lvert \int_c^d f(y)dy - \int_c^d g(y)dy \right\rvert$ where $(f(c), c)$ and $(f(d), d)$ are intersecting points of $x = f(y)$ and $x = g(y)$	

Note :

- If $f(x) \geq g(x)$ in $[c, e]$ and $f(x) \leq g(x)$ in $[e, d]$ where $c < e < d$, then area bounded by the curves is,

$$\int_c^e [f(x) - g(x)]\, dx + \int_e^d [g(x) - f(x)]\, dx$$



- The area of the region lying above the X-axis is positive.
- If the curve lies below the X-axis, then the area is negative. In such cases, we take positive value of area i.e., $A = \left|\int_a^b y\,dx\right|$.

PROBLEMS

Very Short Answer Type

1. Find $\int \sqrt{1+\sin\theta}\,d\theta$.
2. Evaluate : $\int_{-4}^{1} \frac{1}{x}dx$.
3. Find $\int (4x^3+3x^2+2x+4)dx$.
4. Evaluate : $\int \frac{dx}{\sqrt{a+x}}$
5. Find the value of $\int_0^{2\pi} |\sin x|\,dx$.

Long Answer Type-I

6. Evaluate : $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$
7. Using integration, find the area of the region bounded by the parabola $y^2 = 16x$ and the line $x = 4$.
8. Evaluate : $\int_0^{\pi/2} \frac{x}{\sin x+\cos x}dx$
9. If f and g are continuous functions on $[0, a]$ satisfying $f(x) = f(a - x)$ and $g(x) + g(a - x) = 2$, then show that
$$\int_0^a f(x)g(x)dx = \int_0^a f(x)dx$$
10. Evaluate : $\int \frac{x}{(x^2-a^2)(x^2-b^2)}dx$

Long Answer Type-II

11. Evaluate : $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x+\cos x}dx$
12. Using integration, find the area bounded by $x = \frac{1}{2}, x = 2, y = \log_e x$ and $y = 2^x$.
13. Evaluate : $\int_0^{\pi} \frac{x}{1+\cos\alpha\sin x}dx$
14. Evaluate : $\int_0^4 (x+e^{2x})dx$ as the limit of a sum.
15. Sketch the region bounded by the curves $y = \sqrt{5-x^2}$ and $y = |x - 1|$ and find its area.

SOLUTIONS

1. $\int \sqrt{1+\sin\theta}\,d\theta = \int\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)d\theta$

$= 2\sin\frac{\theta}{2} - 2\cos\frac{\theta}{2} + C = 2\left(\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)+C$

2. We have, $\int_{-4}^{1}\frac{1}{x}dx = \left[\log|x|\right]_{-4}^{1}$

$= \log|1| - \log|-4| = 0 - \log 4 = -\log 4$

3. We have, $\int (4x^3+3x^2+2x+4)dx$

$= 4\int x^3dx + 3\int x^2dx + 2\int x\,dx + 4\int x^0dx$

$= 4\cdot\frac{x^4}{4} + 3\cdot\frac{x^3}{3} + 2\cdot\frac{x^2}{2} + 4\cdot x + C$

$= x^4 + x^3 + x^2 + 4x + C$

4. Let $I = \int\frac{dx}{\sqrt{a+x}}$

Put $\sqrt{a+x} = t \Rightarrow \frac{1}{2\sqrt{a+x}}dx = dt$

$\therefore\quad I = \int 2\,dt = 2t + C = 2\sqrt{a+x} + C$

5. We have, $\int_0^{2\pi}|\sin x|dx = \int_0^{\pi}|\sin x|dx + \int_{\pi}^{2\pi}|\sin x|dx$

$= \int_0^{\pi}\sin x\,dx - \int_{\pi}^{2\pi}\sin x\,dx$

$[\because \sin x \ge 0$ for $x \in [0, \pi]$ and $\sin x \le 0$ for $x \in [\pi, 2\pi]]$

$= [-\cos x]_0^{\pi} - [-\cos x]_{\pi}^{2\pi}$

$= -(\cos\pi - \cos 0) + (\cos 2\pi - \cos\pi)$

$= -(-1 - 1) + [1 - (-1)] = 4$

6. Let $I = \int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$ ...(i)

Then, $I = \int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx$

$$\Rightarrow \quad I=\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I=\int_0^{\pi}\frac{(x+\pi-x)\sin x}{1+\cos^2 x}dx=\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$$

$$\Rightarrow \quad I=\frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx \qquad \text{...(iii)}$$

Put $z = \cos x \Rightarrow dz = -\sin x\, dx$

Also, when $x = 0$, $z = 1$ and when $x = \pi$, $z = -1$

$\therefore$ From (iii),

$$I=\frac{\pi}{2}\int_1^{-1}\frac{-dz}{1+z^2}=-\frac{\pi}{2}\int_1^{-1}\frac{1}{1+z^2}dz=-\frac{\pi}{2}[\tan^{-1}z]_1^{-1}$$

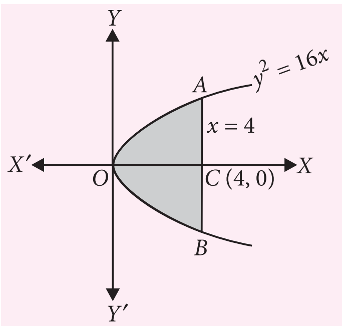
$$=-\frac{\pi}{2}[\tan^{-1}(-1)-\tan^{-1}(1)]=-\frac{\pi}{2}\left[-\frac{\pi}{4}-\frac{\pi}{4}\right]=\frac{\pi^2}{4}$$

7. We have, $y^2 = 16x$ ...(i)

$x = 4$ ...(ii)

Now, required area = area $AOCA$ + area $BOCB$

= 2(area $AOCA$)



$$=2\int_0^4 y\,dx=2\int_0^4\sqrt{16x}\,dx$$

$$=8\int_0^4\sqrt{x}\,dx=8\times\frac{2}{3}\times\left[x^{3/2}\right]_0^4$$

$$=\frac{16}{3}\times(4)^{3/2}=\left(\frac{16}{3}\times 8\right)=\frac{128}{3}\text{ sq. units}$$

8. Let $I=\int\limits_0^{\pi/2}\frac{x}{\sin x+\cos x}dx$ ...(i)

Then, $I=\int\limits_0^{\pi/2}\frac{\frac{\pi}{2}-x}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx$

$$\Rightarrow \quad I=\int\limits_0^{\pi/2}\frac{\frac{\pi}{2}-x}{\sin x+\cos x}dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I=\int\limits_0^{\pi/2}\frac{\frac{\pi}{2}}{\sin x+\cos x}dx\Rightarrow 2I=\frac{\pi}{2}\int\limits_0^{\pi/2}\frac{1}{\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)}$$

$$\Rightarrow \quad I=\frac{1}{2}\cdot\frac{\pi}{2\sqrt{2}}\int_0^{\pi/2}\sec\left(x-\frac{\pi}{4}\right)dx$$

$$=\frac{\pi}{4\sqrt{2}}\left[\log\left|\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right|\right]_0^{\pi/2}$$

$$=\frac{\pi}{4\sqrt{2}}\left[\log\left|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right|-\log\left|\sec\frac{\pi}{4}-\tan\frac{\pi}{4}\right|\right]$$

$$=\frac{\pi}{4\sqrt{2}}[\log(\sqrt{2}+1)-\log(\sqrt{2}-1)]$$

$$=\frac{\pi}{4\sqrt{2}}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)=\frac{\pi}{4\sqrt{2}}\log(\sqrt{2}+1)^2$$

$$=\frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$$

9. Let $I=\int_0^a f(x)g(x)dx$ ...(i)

Then, $I=\int_0^a f(a-x)g(a-x)dx$

$$=\int_0^a f(x)[2-g(x)]dx$$

$$=2\int_0^a f(x)dx-\int_0^a f(x)g(x)dx$$

$$=2\int_0^a f(x)dx-I$$

$$\therefore \quad 2I=2\int_0^a f(x)dx \Rightarrow I=\int_0^a f(x)dx$$

10. Let $I=\int\frac{x}{(x^2-a^2)(x^2-b^2)}dx$

Put $z = x^2 \Rightarrow 2x dx = dz$

$$\therefore \quad I=\frac{1}{2}\int\frac{dz}{(z-a^2)(z-b^2)} \qquad \text{...(i)}$$

Let $\frac{1}{(z-a^2)(z-b^2)}=\frac{A}{z-a^2}+\frac{B}{z-b^2}$

$\Rightarrow \quad A(z - b^2) + B(z - a^2) = 1$

Putting $z = a^2$, we get $A=\frac{1}{a^2-b^2}$

Putting $z = b^2$, we get $B=-\frac{1}{a^2-b^2}$

$\therefore$ From (i),

$$I=\frac{1}{2}\cdot\frac{1}{(a^2-b^2)}\int\left(\frac{1}{z-a^2}-\frac{1}{z-b^2}\right)dz$$

$$=\frac{1}{2(a^2-b^2)}\left[\log|z-a^2|-\log|z-b^2|\right]+C$$

$$= \frac{1}{2(a^2-b^2)} \log\left|\frac{z-a^2}{z-b^2}\right| + C$$

$$= \frac{1}{2(a^2-b^2)} \log\left|\frac{x^2-a^2}{x^2-b^2}\right| + C$$

11. Let $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$ ...(i)

Then, $I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$

$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} dx$ ...(ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{1}{\frac{2\tan(x/2)}{1+\tan^2(x/2)} + \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}} dx$$

$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{\sec^2 \frac{x}{2}}{2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}} dx$ ...(iii)

Put $\tan\frac{x}{2} = z$, then

$$\sec^2\frac{x}{2} \cdot \frac{1}{2} dx = dz \Rightarrow \sec^2\frac{x}{2} dx = 2dz$$

Also when $x = 0$, $z = 0$ and when $x = \frac{\pi}{2}$, $z = 1$

Now from (iii),

$$2I = \int_0^1 \frac{2dz}{2z+1-z^2} = 2\int_0^1 \frac{1}{(\sqrt{2})^2 - (z-1)^2} dz$$

$$= 2 \times \frac{1}{2\sqrt{2}} \left[\log\left|\frac{\sqrt{2}+z-1}{\sqrt{2}-z+1}\right|\right]_0^1$$

$$= \frac{1}{\sqrt{2}} \left\{0 - \log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right\}$$

$$= -\frac{1}{\sqrt{2}} \log(\sqrt{2}-1)^2 = -\frac{2}{\sqrt{2}} \log(\sqrt{2}-1)$$

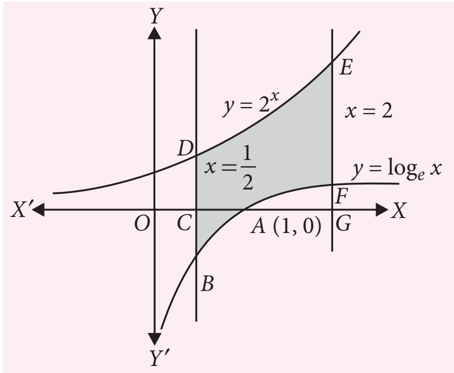
$$\therefore \quad I = -\frac{1}{\sqrt{2}} \log(\sqrt{2}-1) = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$$

12. Given curves are $y = \log_e x$ ...(i)

and $y = 2^x$ ...(ii)

Given lines are $x = \frac{1}{2}$ and $x = 2$

Hence shaded part is the region bounded by curves (i) and (ii) and the lines $x = \frac{1}{2}$ and $x = 2$.



Required area

$$= \int_{1/2}^2 (2^x - \log x) dx = \left[\frac{2^x}{\log 2} - (x\log x - x)\right]_{1/2}^2$$

$$= \left(\frac{4}{\log 2} - 2\log 2 + 2\right) - \left(\frac{\sqrt{2}}{\log 2} - \frac{1}{2}\log\frac{1}{2} + \frac{1}{2}\right)$$

$$= \left(\frac{4-\sqrt{2}}{\log 2} + \frac{3}{2} - \frac{5}{2}\log 2\right) \text{ sq. units}$$

13. Let $I = \int_0^{\pi} \frac{x}{1+\cos\alpha \sin x} dx$ ...(i)

Then, $I = \int_0^{\pi} \frac{\pi - x}{1+\cos\alpha \sin(\pi - x)} dx$

$= \int_0^{\pi} \frac{\pi - x}{1+\cos\alpha \sin x} dx$ ...(ii)

Adding (i) and (ii), we get

$$2I = \pi\int_0^{\pi} \frac{dx}{1+\cos\alpha \sin x}$$

$= 2\int_0^{\pi/2} \frac{dx}{1+a\sin x}$, where $a = \cos\alpha$

$$= 2\pi\int_0^{\pi/2} \frac{dx}{1 + a\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$= 2\pi\int_0^{\pi/2} \frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2} + 2a\tan\frac{x}{2} + 1} dx = 2\pi\int_0^1 \frac{2}{z^2 + 2az + 1} dz$$

$$\left[\text{Put } z = \tan\frac{x}{2} \Rightarrow dz = \frac{1}{2}\sec^2\frac{x}{2} dx\right]$$

$$= 4\pi\int_0^1 \frac{dz}{(z+a)^2+1-a^2} = 4\pi\int_0^1 \frac{dz}{(z+\cos\alpha)^2+\sin^2\alpha}$$

$[\because a = \cos\alpha]$

$$= \frac{4\pi}{\sin\alpha}\left[\tan^{-1}\frac{z+\cos\alpha}{\sin\alpha}\right]_0^1$$

$$= \frac{4\pi}{\sin\alpha}\left[\tan^{-1}\left(\cot\frac{\alpha}{2}\right) - \tan^{-1}(\cot\alpha)\right]$$

$$= \frac{4\pi}{\sin\alpha}\left[\tan^{-1}\tan\left(\frac{\pi}{2}-\frac{\alpha}{2}\right) - \tan^{-1}\tan\left(\frac{\pi}{2}-\alpha\right)\right]$$

$$= \frac{4\pi}{\sin\alpha}\left[\left(\frac{\pi}{2}-\frac{\alpha}{2}\right) - \left(\frac{\pi}{2}-\alpha\right)\right] = \frac{2\pi\alpha}{\sin\alpha} \quad \therefore \quad I = \frac{\pi\alpha}{\sin\alpha}$$

14. We have, $f(x) = x + e^{2x}$, $a = 0$, $b = 4$

$nh = b - a = 4 - 0 = 4$

Now

$$\int_0^4 x + e^{2x} = \lim_{h\to 0} h[f(0) + f(0+h) + \ldots f\{0+(n-1)h\}]$$

$$= \lim_{h\to 0} h[1 + (h + e^{2h}) + 2h + e^{2(2h)} + \ldots (n-1)h + e^{2(n-1)h}]$$

$$= \lim_{h\to 0} h[(h + 2h + \ldots + (n-1)h) + (1 + e^{2h} + e^{2(2h)} \ldots e^{2(n-1)h})]$$

$$= \lim_{h\to 0} h\left[h\frac{(n-1)n}{2} + \frac{e^{2nh}-1}{e^{2h}-1}\right]$$

$$= \lim_{h\to 0}\left[\frac{nh(nh-h)}{2} + h\frac{(e^{2nh}-1)}{\left(\frac{e^{2h}-1}{2h}\right)2h}\right]$$

$$= \lim_{h\to 0}\left[\frac{4(4-h)}{2} + \frac{(e^{2\times 4}-1)}{\left(\frac{e^{2h}-1}{2h}\right)\times 2}\right]$$

$$= 8 + \frac{e^8-1}{2} \qquad \left[\because \lim_{h\to 0}\frac{e^{2h}-1}{2h} = 1\right]$$

$$= \frac{1}{2}(16 + e^8 - 1) = \frac{1}{2}(15 + e^8)$$

15. Given curves are $y = \sqrt{5-x^2}$ ...(i)

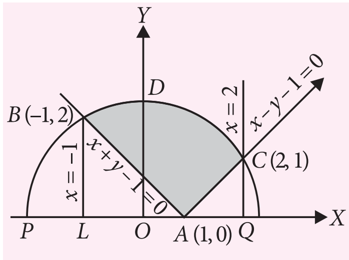
and $y = |x - 1|$ ...(ii)

Curve (i) is the part of the circle $x^2 + y^2 = 5$ above x-axis.

$$\text{and } y = |x-1| = \begin{cases} x-1, & x \ge 1 \\ -x+1, & x < 1 \end{cases}$$

From (i) and (ii), we have $\sqrt{5-x^2} = |x-1|$

$\therefore \quad 5 - x^2 = (x - 1)^2 \Rightarrow x = -1, 2$



Required area

$$= \int_{-1}^2 \sqrt{5-x^2}\,dx - \int_{-1}^2 |x-1|\,dx$$

Now, $\int_{-1}^2 \sqrt{5-x^2}\,dx$

$$= \left[\frac{1}{2}x\sqrt{5-x^2} + \frac{1}{2}\cdot 5\sin^{-1}\frac{x}{\sqrt{5}}\right]_{-1}^2$$

$$= \left[\left(1 + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}\right) - \left(-1 + \frac{5}{2}\sin^{-1}\left(\frac{-1}{\sqrt{5}}\right)\right)\right]$$

$$= 2 + \frac{5}{2}\left[\sin^{-1}\frac{2}{\sqrt{5}} - \sin^{-1}\left(-\frac{1}{\sqrt{5}}\right)\right]$$

$$= 2 + \frac{5}{2}\left[\sin^{-1}\frac{2}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{5}}\right]$$

$$= 2 + \frac{5}{2}\left[\sin^{-1}\left(\frac{2}{\sqrt{5}}\cdot\sqrt{1-\frac{1}{5}} + \frac{1}{\sqrt{5}}\sqrt{1-\frac{4}{5}}\right)\right]$$

$$= 2 + \frac{5}{2}\sin^{-1}(1) = 2 + \frac{5}{2}\cdot\frac{\pi}{2} = \frac{5\pi}{4} + 2$$

Again, $\int_{-1}^2 |x-1|\,dx = \int_{-1}^1 |x-1|\,dx + \int_1^2 |x-1|\,dx$

$$= -\int_{-1}^1 (x-1)\,dx + \int_1^2 (x-1)\,dx$$

$$= \left[x - \frac{x^2}{2}\right]_{-1}^1 + \left[\frac{x^2}{2} - x\right]_1^2 = \frac{5}{2}$$

So, required area $= \dfrac{5\pi}{4} + 2 - \dfrac{5}{2} = \left(\dfrac{5\pi}{4} - \dfrac{1}{2}\right)$ sq. units.

MPP-3 CLASS XI ANSWER KEY

1.	(b)	**2.**	(c)	**3.**	(d)	**4.**	(c)	**5.**	(a)
6.	(c)	**7.**	(c,d)	**8.**	(b,c)	**9.**	(a,d)	**10.**	(a,b)
11.	(a,d)	**12.**	(a,c)	**13.**	(d)	**14.**	(d)	**15.**	(c)
16.	(d)	**17.**	(4)	**18.**	(7)	**19.**	(2)	**20.**	(3)

MPP-3 MONTHLY Practice Problems

Class XII

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Continuity & Differentiability

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. Let $f(x)=\dfrac{\tan x\log x}{1-\cos 4x}$, then $f(x)$ is discontinuous at

(a) $\left\{\dfrac{n\pi}{2};n\in Q\right\}$ (b) $\{(2n+1)\pi/2:n\in Z\}$

(c) $\left\{\dfrac{n\pi}{2}:n\in N\right\}\cup(-\infty,0)$

(d) none of these

2. Let $f: R\to R$ be a function defined by $f(x) = \max\{x, x^3\}$. The set of all points where $f(x)$ is not differentiable is

(a) $\{-1, 1\}$ (b) $\{-1, 0\}$ (c) $\{0, 1\}$ (d) $\{-1, 0, 1\}$

3. If $y=\log_{10}x+\log_x 10+\log_x x+\log_{10}10$, then $\left(\dfrac{dy}{dx}\right)_{x=10}$ is equal to

(a) 2 (b) 1 (c) 0 (d) −1

4. $y=(x+\sqrt{1+x^2})^n$ then $(1+x^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}$ is

(a) n^2y (b) $-n^2y$ (c) $-y$ (d) $2n^2y$

5. If $g(x)=\begin{cases}[f(x)], & x\in\left(0,\dfrac{\pi}{2}\right)\cup\left(\dfrac{\pi}{2},\pi\right)\\ 3, & x=\dfrac{\pi}{2}\end{cases}$

where $[x]$ denotes the greatest integer function and

$f(x)=\dfrac{2(\sin x-\sin^n x)+|\sin x-\sin^n x|}{2(\sin x-\sin^n x)-|\sin x-\sin^n x|}, n\in R$, then

(a) $g(x)$ is continuous and differentiable at $x = \pi/2$, when $0 < n < 1$

(b) $g(x)$ is continuous and differentiable at $x = \pi/2$, when $n > 1$

(c) $g(x)$ is continuous but not differentiable at $x = \pi/2$, when $0 < n < 1$

(d) $g(x)$ is continuous but not differentiable at $x = \pi/2$, when $n > 1$

6. Let $f(x)=\begin{vmatrix}x^3 & \sin x & \cos x\\ 6 & -1 & 0\\ p & p^2 & p^3\end{vmatrix}$ where p is a constant.

Then $\dfrac{d^3}{dx^3}(f(x))$ at $x=0$ is

(a) p (b) $p+p^2$

(c) $p+p^3$ (d) Independent of p.

One or More Than One Option Correct Type

7. The function $f(x)=\begin{cases}|x-3| & ,x\ge 1\\ \dfrac{x^2}{4}-\dfrac{3x}{2}+\dfrac{13}{4} & ,x<1\end{cases}$.

(a) continuous at $x = 1$ (b) differentiable at $x = 1$

(c) continuous at $x = 3$ (d) differentiable at $x = 3$

8. If $f(x) = \min\{1, x^2, x^3\}$, then

(a) $f(x)$ is continuous $\forall\ x\in R$

(b) $f(x)$ is not differentiable $\forall\ x\in R$

(c) $f(x)$ is not differentiable at two points

(d) None of these.

9. The function $\begin{cases}| \quad - |(\ ) & \ge\\ \sin\left(\dfrac{\pi}{\ }\right) & <\end{cases}$

([.] denotes the greatest integer function)

(a) is differentiable at $x = 0$

(b) is continuous but not differentiable at $x = 1$

(c) is continuous at $x = 0$

(d) is continuous but not differentiable at $x = 3/2$

10. If $f(x)=\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$, then

(a) f is derivable for all x, with $|x| < 1$

(b) f is not derivable at $x = 1$

(c) f is not derivable at $x = -1$
(d) f is derivable for all x, with $|x| > 1$

11. If $y = \dfrac{\sqrt{(1+t^2)} - \sqrt{(1-t^2)}}{\sqrt{(1+t^2)} + \sqrt{(1-t^2)}}$ and $x = \sqrt{(1-t^4)}$, then $\dfrac{dy}{dx}$ equal to

(a) $\dfrac{-1}{t^2\left\{1+\sqrt{(1-t^4)}\right\}}$ (b) $\dfrac{\left\{\sqrt{(1-t^4)}-1\right\}}{t^6}$

(c) $\dfrac{1}{t^2\left\{1+\sqrt{(1-t^4)}\right\}}$ (d) $\dfrac{1-\sqrt{(1-t^4)}}{t^6}$

12. Let $f(x) = \dfrac{1}{[\sin x]}$, ([.] denotes the greatest integer function) then
(a) domain of $f(x)$ is $(2n\pi + \pi, 2n\pi + 2\pi) \cup \{2n\pi + \pi/2\}$, where $n \in I$
(b) $f(x)$ is continuous, when $x \in (2n\pi + \pi, 2n\pi + 2\pi)$, where $n \in I$
(c) $f(x)$ is differentiable at $x = \pi/2$
(d) none of these

13. If $F(x) = f(x)g(x)$ and $f'(x)g'(x) = c$, then

(a) $F' = c\left[\dfrac{f}{f'} + \dfrac{g}{g'}\right]$ (b) $\dfrac{F''}{F} = \dfrac{f''}{f} + \dfrac{g''}{g} + \dfrac{2c}{fg}$

(c) $\dfrac{F'''}{F} = \dfrac{f'''}{f} + \dfrac{g''}{g}$ (d) $\dfrac{F'''}{F''} = \dfrac{f'''}{f''} + \dfrac{g'''}{g''}$

Comprehension Type

If $D^* f(x) = \lim\limits_{h\to 0} \dfrac{f^2(x+h) - f^2(x)}{h}$ here $f^2(x) = \{f(x)\}^2$

14. If $u = f(x)$, $v = g(x)$, then the value of $D^*(u \cdot v)$ is
(a) $(D^* u)\, v + (D^* v) u$ (b) $u^2(D^* v) + v^2(D^* u)$
(c) $D^* u + D^* v$ (d) $uv(D^*(u + v)$

15. If $u = f(x)$, $v = g(x)$ then the value of $D^*\left\{\dfrac{u}{v}\right\}$ is

(a) $\dfrac{u^2(D^* v) - v^2(D^* u)}{v^4}$ (b) $\dfrac{u(D^* v) - v(D^* u)}{v^2}$

(c) $\dfrac{v^2(D^* u) - u^2(D^* v)}{v^4}$ (d) $\dfrac{u(D^* u) - u(D^* v)}{v^2}$

Matrix Match Type

16. Match the columns:

	Column I		Column II
P	$y = \sin^{-1}(3x-4x^3)$, then $\dfrac{dy}{dx}$ is	1	$\dfrac{3}{1+x^2}, x \in \left(-\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$
Q	$y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ then $\dfrac{dy}{dx}$ is	2	$\dfrac{-2}{1+x^2}, x < 0$
R	$y = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$, then $\dfrac{dy}{dx}$ is	3	$\dfrac{3}{\sqrt{(1-x^2)}}, x \in \left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$

	P	Q	R
(a)	2	1	3
(b)	3	1	2
(c)	3	2	1
(d)	1	2	3

Integer Answer Type

17. If $f(x)$ is a continuous function $\forall\ x \in R$ and the range of $f(x)$ is $\left(2, \sqrt{26}\right)$ and $g(x) = \left[\dfrac{f(x)}{c}\right]$ is continuous $\forall\ x \in R$, then the least positive integral value of c is (where [.] denotes the greatest integer function)

18. If $y = \tan^{-1}\left(\dfrac{\log_e(e/x^2)}{\log_e(ex^2)}\right) + \tan^{-1}\left(\dfrac{3+2\log_e x}{1-6\log_e x}\right)$, then $\dfrac{d^2y}{dx^2}$ is

19. If $y = 2\sin^{-1}\left(\dfrac{x-2}{\sqrt{6}}\right) - \sqrt{\left(2+4x-x^2\right)}$, then at $x = 2$ the value of $2\sqrt{\dfrac{3}{2}}\dfrac{dy}{dx}$ must be

20. If $f(x) = \begin{cases} \dfrac{8^x - 4^x - 2^x + 1^x}{x^2}, & x > 0 \\ e^x \sin x + \pi x + \lambda \ln 4, & x \le 0 \end{cases}$ is continuous at $x = 0$, then the value of $\dfrac{e^\lambda}{2}$ must be

◈◈

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ……
No. of questions correct ……
Marks scored in percentage ……

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.

During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

PROBLEM Set 165

JEE MAIN

1. Let P be a point on the parabola $y = x^2$ with focus S. If $A = (1, 2)$, then the minimum value of $(PS + PA)$ is

(a) $\frac{5}{4}$ (b) $\frac{7}{4}$ (c) $\frac{9}{4}$ (d) $\frac{5}{2}$

2. A five-digit number is formed with the digits 0, 1, 2, 3, 4, 5 without repetition. The probability that it is divisible by 11 is

(a) $\frac{2}{25}$ (b) $\frac{2}{50}$

(c) $\frac{3}{50}$ (d) $\frac{3}{25}$

3. If $\sum_{r=1}^{10} \frac{1}{(2r-1)(2r+1)(2r+3)} = \frac{m}{n}$, reduced fraction, then the sum of digits of $(m + n)$ is

(a) 9 (b) 10 (c) 11 (d) 13

4. If the equation $x^2 + ax + 6a = 0$ has integer roots, then the number of values of a is

(a) 2 (b) 4 (c) 6 (d) 10

5. If $z = a + ib$, $a, b \in R$, $b \neq 0$ and $|z| = 1$, then $z = \frac{c+i}{c-i}$, where $c =$

(a) $\frac{a}{b}$ (b) $\frac{a-1}{b}$

(c) $\frac{a+1}{b}$ (d) $\frac{a+1}{b+1}$

JEE ADVANCED

6. $\tan^3\theta + \cot^3\theta = 12 + 8\operatorname{cosec}^3 2\theta$ if $\theta =$

(a) $\frac{7\pi}{12}$ (b) $\frac{11\pi}{12}$

(c) $\frac{19\pi}{12}$ (d) $\frac{23\pi}{12}$

COMPREHENSION

Let $\frac{dy}{dx} = \frac{1}{x + y}$, $y(0) = 0$. Then

7. At $y = \ln 3$, $\frac{dy}{dx} =$

(a) 1 (b) -1 (c) 1/2 (d) -2

8. $\int_0^1 x\,dy =$

(a) $\frac{1}{2}$ (b) $\frac{e}{2}$ (c) $e - \frac{1}{2}$ (d) $e - \frac{5}{2}$

INTEGER MATCH

9. Let $a_1, a_2, a_3, ..., a_{10}$ be a permutation of the set $\{1, 2, 3, ..., 10\}$ such that the sequence a_i decreases first and then increases like 8, 6, 4, 1, 2, 3, 5, 7, 9, 10. If N is the number of such permutations, then the sum of digits of N is

MATRIX MATCH

10. Match the following columns.

	Column I	Column II	
(P)	If 4 dice are rolled once, the number of ways of getting the sum 10 is	1.	84
(Q)	If a, b, c, d are odd positive integers, the number of solutions of the equation $a + b + c + d = 16$ is	2.	55
(R)	The coefficient of x^2y in the expansion of $(1 + x + 2y)^5$ is	3.	60
(S)	If $C_r = \binom{10}{r}$, then $\sum_{r=1}^{10} \frac{r \cdot C_r}{C_{(r-1)}}$ is	4.	80

	P	Q	R	S
(a)	2	3	1	4
(b)	4	1	3	2
(c)	4	3	2	1
(d)	2	1	3	4

See Solution Set of Maths Musing 164 on page no. 85

MATH archives

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of JEE (Main and Advanced) and other PETs. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for competitions. In every issue of MT, challenging problems are offered with detailed solutions. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. Let $p=\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right)$

and $q=\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$ then

(a) $p = q$ (b) $2p = q$
(c) $p = 2q$ (d) $p + q = 1/4$

2. Let $2x - 3y = 0$ be a given line and $P(\sin\theta, 0)$ and $Q(0, \cos\theta)$ be two points. Then P and Q lie on the same side of the given line if θ lies in the

(a) 1st quadrant (b) 2nd quadrant
(c) 3rd quadrant (d) 4th quadrant

3. If $\sin^4 x\cos 3x = \sum_{k=0}^{n} a_k \cos kx$. Then the value of n is

(a) 5 (b) 6
(c) 7 (d) none of these

4. Let $f: R \to R$ be a function defined by,

$f(x)=\dfrac{2x^2-x+5}{7x^2+2x+10}$ then f is

(a) injective but not surjective
(b) surjective but not injective
(c) injective as well as surjective
(d) neither injective nor surjective

5. If α and β are two distinct roots of the equation $a\tan x + b\sec x = c$, then $\tan(\alpha+\beta)$ is equal to

(a) $\dfrac{ac}{a^2+c^2}$ (b) $\dfrac{2ac}{a^2+c^2}$

(c) $\dfrac{2ac}{a^2-c^2}$ (d) none of these

6. If $g(\theta)=\sin^2\theta+\sin^2\left(\theta+\frac{\pi}{3}\right)+\cos\theta.\cos\left(\theta+\frac{\pi}{3}\right)$ and $f\left(\frac{5}{4}\right)=1$ then find $(fog)(x)$.

7. If $\{x\}$ and $[x]$ denote the fractional and integral parts of a real number x respectively, then solve $2x + \{x + 1\} = 4[x + 1] - 6$.

8. Find the domain of the following functions

(a) $\dfrac{\cos^{-1}x}{[x]}$ where $[\cdot]$ denotes the greatest integer function.

(b) $\dfrac{1}{x}+2^{\sin^{-1}x}+\dfrac{1}{\sqrt{x-2}}$

9. Evaluate $\underset{x\to 0}{Lt}\dfrac{1-\cos x.\sqrt{\cos 2x}}{x^2}$ without using L' Hospital Rule.

10. If $f(x)=\begin{cases}\left(1+|\sin x|\right)^{\frac{p}{|\sin x|}} & ; -\frac{\pi}{6}<x<0\\ q & ; \quad x=0\\ e^{\frac{\tan 3x}{\tan 5x}} & ; \quad 0<x<\frac{\pi}{6}\end{cases}$

is continuous at $x = 0$. Find the values of p and q.

By : **Prof. Shyam Bhushan,** Director, Narayana IIT Academy, Jamshedpur. Mob. : 09334870021

SOLUTIONS

1. **(b) :** $p = \left(\sin\frac{\pi}{10}\sin\frac{3\pi}{10}\right)^2 = \frac{1}{16}$

and $q = \left(\sin\frac{\pi}{8}\sin\frac{3\pi}{8}\right)^2 = \frac{1}{8}$

Hence, $q = 2p$

2. **(b) :** $L \equiv 2x - 3y = 0\,;\, L(P)\cdot L(Q) > 0$

i.e., $L(\sin\theta, 0)\cdot L(0, \cos\theta) > 0$

$\Rightarrow \sin\theta\cdot\cos\theta < 0$

$\Rightarrow \sin 2\theta < 0$

$\therefore \frac{\pi}{2} < \theta < \pi$

3. **(c)** **4.** **(d)**

5. **(c) :** $(a^2 - b^2)\tan^2 x - 2ac\tan x + (c^2 - b^2) = 0$

$\therefore \tan\alpha + \tan\beta = \frac{2ac}{a^2 - b^2}$

$\tan\alpha\tan\beta = \frac{c^2 - b^2}{a^2 - b^2}$

Hence, $\tan(\alpha+\beta) = \frac{2ac}{a^2 - c^2}$

6. $g(\theta) = \sin^2\theta + \sin^2\left(\theta + \frac{\pi}{3}\right) + \cos\theta.\cos\left(\theta+\frac{\pi}{3}\right)$

$$= \frac{1}{2}\left[1 - \cos 2\theta + 1 - \cos\left(2\theta + \frac{2\pi}{3}\right) + \cos\left(2\theta + \frac{\pi}{3}\right) + \cos\frac{\pi}{3}\right]$$

$$= \frac{1}{2}\left[\frac{5}{2} - 2\cos\left(2\theta + \frac{\pi}{3}\right)\cos\frac{\pi}{3} + \cos\left(2\theta + \frac{\pi}{3}\right)\right] = \frac{5}{4}\,\forall\theta$$

$\therefore$ $(fog)(x) = f[g(x)]$

$= f\left(\frac{5}{4}\right)$ $\quad\left(\because f\left(\frac{5}{4}\right) = 1 \text{ given}\right)$

$= 1$

7. $2x + \{x + 1\} = 4[x + 1] - 6$

$\Rightarrow 2x + x + 1 - [x] - 1 = 4[x] - 2$

$[\because [x + n] = [x] + n\,;\, n \in I]$

$\Rightarrow 5[x] = 3x + 2$ ...(1)

$= 3([x] + \{x\}) + 2$

$\Rightarrow 3\{x\} = 2[x] - 2$ ...(2)

Now, $0 \le \{x\} < 1$

$\Rightarrow 0 \le 3\{x\} < 3$

$\Rightarrow 0 \le 2[x] - 2 < 3$

$\Rightarrow 2 \le 2[x] < 5$

$\Rightarrow 1 \le [x] < \frac{5}{2} \Rightarrow [x] = 1, 2$

$\Rightarrow [x] = 1$

$\Rightarrow x = 1$

and $[x] = 2 \Rightarrow x = \frac{8}{3}$ [From (1)]

8. (a) $D_{\cos^{-1}x} = [-1, 1]$

$D_{[x]} = R$

$\therefore D_{\frac{\cos^{-1}x}{[x]}} = [-1,1] \cap (R - \{x \mid [x] = 0\})$

$= [-1, 0) \cup \{1\}$

(b) $D_{\frac{1}{x} + 2^{\sin^{-1}x} + \frac{1}{\sqrt{x-2}}} = (R - \{0\}) \cap ([-1,1]) \cap (2,\infty) = \phi$

$\therefore$ $f(x)$ is not defined for any $x \in R$

9. $\underset{x\to 0}{Lt}\frac{1 - \cos x\sqrt{\cos 2x}}{x^2}$

$$= \underset{x\to 0}{Lt}\frac{1 - \cos^2 x\cos 2x}{x^2\left(1 + \cos x\sqrt{\cos 2x}\right)}$$

$$= \underset{x\to 0}{Lt}\frac{1 - \cos^2 x\left(2\cos^2 x - 1\right)}{x^2\left(1 + \cos x\sqrt{\cos 2x}\right)}$$

$$= -\underset{x\to 0}{Lt}\frac{2\cos^4 x - \cos^2 x - 1}{x^2\left(1 + \cos x\sqrt{\cos 2x}\right)}$$

$$= \underset{x\to 0}{Lt}\left(\frac{2\cos^2 x + 1}{1 + \cos x\sqrt{\cos 2x}}\right)\left(\frac{\sin^2 x}{x^2}\right) = \frac{3}{2}$$

10. Since f is continuous at $x = 0$

$\therefore \underset{x\to 0^-}{Lt} f(x) = f(0) = \underset{x\to 0^+}{Lt} f(x)$

$\Rightarrow e^p = q = e^{\frac{3}{5}} \Rightarrow p = \frac{3}{5}, q = e^{\frac{3}{5}}$

WB JEE MOCK TEST PAPER

Series-2

The entire syllabus of Mathematics of WB-JEE is being divided into eight units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issues. The syllabus for module break-up is given bellow:

Unit No.	Topic	Syllabus In Details
UNIT NO. 2	Complex Numbers	Algebra of complex numbers, addition, multiplication, conjugation, polar representation, properties of modulus and principal argument, triangle inequality, square root of a complex number cube roots of unity, geometric interpretations.
	Quadratic Equations	Quadratic equations in real and complex system and their solutions. Quadratic equations with real coefficients, relation between roots and coefficients, formation of quadratic equations with given roots, symmetric functions of roots.
	Trigonometry	Compound angles, Transformation of sum & product, Multiple & sub-multiple angles.
	Co-ordinate Geometry-2D	Straight lines: Equation of a straight line in various forms, angle between two lines, distance of a point from a line, lines through the point of intersection of two given lines, equation of the bisector of the angle between two lines, concurrency of lines, centroid, orthocenter, incentre and circumcentre of a triangle.

Time : 1 hr 15 min **Full marks : 50**

CATEGORY-I

For each correct answer one mark will be awarded, whereas, for each wrong answer, 25% of total marks (1/4) will be deducted. If candidates mark more than one answer, negative marking will be done.

1. If $z = x + iy$ and $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$, then the locus of the point (x, y) is
(a) an ellipse
(b) a straight line
(c) a circle with centre (0, 1)
(d) a circle with centre (1, 0)

2. If ω be the imaginary cube root of 1, the value of $\frac{7+11\omega+13\omega^2}{13+7\omega+11\omega^2}+\frac{7+11\omega+13\omega^2}{11+13\omega+7\omega^2}$ will be
(a) 2 (b) 3 (c) 0 (d) –1

3. Two sides of a rhombus are along the lines, $x - y + 1 = 0$ and $7x - y - 5 = 0$. If its diagonals intersects at $(-1, -2)$, then which one of the following is a vertex of this rhombus?
(a) $(-3, -9)$ (b) $(-3, -8)$
(c) $\left(\frac{1}{3}, -\frac{8}{3}\right)$ (d) $\left(-\frac{10}{3}, -\frac{7}{3}\right)$

4. If $y = \frac{1}{2}\sin^2\theta + \frac{1}{3}\cos^2\theta$, then
(a) $\frac{1}{3} \le y \le \frac{1}{2}$ (b) $y \ge \frac{1}{2}$
(c) $2 \le y \le 3$ (d) $-\frac{\sqrt{13}}{6} \le y \le \frac{\sqrt{13}}{6}$

5. A triangle with vertices (4, 0), (–1, –1), (3, 5) is
(a) isosceles and right-angled
(b) isosceles but not right-angled
(c) right-angled but not isoceles
(d) neither right-angled nor isosceles.

By : Sankar Ghosh, HOD(Math), Takshyashila. Mob : 09831244397.

6. The number of points on the line $x + y = 4$, which are unit distance apart from the line $2x + 2y = 5$ is
(a) 0 (b) 1 (c) 2 (d) infinity

7. If for a variable line $\frac{x}{a}+\frac{y}{b}=1$, the condition $a^{-2} + b^{-2} = c^{-2}$ (c is a constant) is satisfied, then the locus of foot of the perpendicular drawn from the origin to this line is
(a) $x^2 + y^2 = \frac{c^2}{2}$ (b) $x^2 + y^2 = 2c^2$
(c) $x^2 + y^2 = c^2$ (d) $x^2 - y^2 = c^2$

8. The triangle formed by the lines $x^2 - 3y^2 = 0$ and $x = 4$, is
(a) isosceles (b) equilateral
(c) right-angled (d) none of these

9. If $\sqrt[3]{x+iy}=a+ib$, then $\frac{x}{a}+\frac{y}{b}=2\lambda$, where the value of λ is
(a) $a^2 - b^2$ (b) $2(a^2 - b^2)$
(c) $4(a^2 - b^2)$ (d) none of these

10. If $(1 + i)(1 + 2i)(1 + 3i) \ldots (1 + ni) = a + ib$, then $2\cdot5\cdot10 \ldots (1 + n^2)$ is equal to
(a) $a^2 + b^2$ (b) $\sqrt{a^2+b^2}$
(c) $\sqrt{a^2-b^2}$ (d) $a^2 - b^2$

11. The harmonic mean of the roots of the equation $(5+\sqrt{2})x^2-(4+\sqrt{5})x+8+2\sqrt{5}=0$ is
(a) 2 (b) 4 (c) 6 (d) 8

12. The quadratic equation whose roots are sin 18° and cos 36°, is
(a) $2x^2-\sqrt{5}x+1=0$ (b) $4x^2-2\sqrt{5}x+1=0$
(c) $2x^2+\sqrt{5}x+1=0$ (d) $4x^2+\sqrt{5}x+1=0$

13. The value of $\sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}+\sin^2\frac{5\pi}{8}+\sin^2\frac{7\pi}{8}$ is equal to
(a) 2 (b) 1/8 (c) 1/4 (d) 1

14. The value of $\frac{\sqrt{3}}{\sin 15^\circ}-\frac{1}{\cos 15^\circ}$ is equal to
(a) $4\sqrt{2}$ (b) $2\sqrt{2}$ (c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$

15. If $\tan\theta=\frac{1}{2}$ and $\tan\phi=\frac{1}{3}$, then $\tan(2\theta+\phi)=$
(a) 3/4 (b) 4/3 (c) 1/3 (d) 3

16. The value of
$$\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$$ is
(a) 1/2 (b) $\cos\frac{\pi}{8}$
(c) 1/8 (d) $\frac{1+\sqrt{2}}{2\sqrt{2}}$

17. If y-intercept of the line $4x - ay = 8$ is thrice its x-intercept, then the value of a is equal to
(a) 3/4 (b) 4/3 (c) –3/4 (d) –4/3

18. The equation of one of the straight lines passing through the point (0, 1) and is at a distance of 3/5 units from the origin is
(a) $4x + 3y = 3$ (b) $-x + y = 1$
(c) $5x + 4y = 4$ (d) $-5x + 4y = 4$

19. The image of the origin with respect to the line $4x + 3y = 25$, is
(a) (4, 3) (b) (3, 4)
(c) (6, 8) (d) (8, 6)

20. The principal argument of the complex number
$$z=\frac{1+\sin\frac{\pi}{3}+i\cos\frac{\pi}{3}}{1+\sin\frac{\pi}{3}-i\cos\frac{\pi}{3}}$$ is
(a) $\pi/3$ (b) $\pi/6$
(c) $2\pi/3$ (d) $\pi/2$

21. If $\frac{(1+i)(2+3i)(3-4i)}{(2-3i)(1-i)(3+4i)}=a+ib$, then $a^2+b^2=$
(a) 132 (b) 25 (c) 144 (d) 1

22. Let z, w be two non-zero complex numbers. If $\overline{z}+i\overline{w}=0$ and arg $(zw) = \pi$, then arg z =
(a) π (b) $\pi/2$ (c) $\pi/4$ (d) $\pi/6$

23. If $z=\frac{2-i}{i}$, then $\text{Re}(z^2)+\text{Im}(z^2)=$
(a) 1 (b) –1 (c) 2 (d) 7

24. If $|z + 1| < |z - 1|$, then z lies
(a) on the x-axis (b) on the y-axis
(c) in the region $x < 0$
(d) in the region $y > 0$

25. If $\left|z-\frac{3}{z}\right|=2$, then the greatest value of $|z|$ is
(a) 1 (b) 2 (c) 3 (d) 4

26. If the roots of the quadratic equation $mx^2 - nx + k = 0$ are tan 33° and tan 12°, then the value of $\frac{2m+n+k}{m} =$

(a) 0 (b) 1 (c) 2 (d) 3

27. If α and β are the roots of $4x^2 + 2x - 1 = 0$, then $\beta =$

(a) $-\frac{1}{4\alpha}$ (b) $-\frac{1}{2\alpha}$ (c) $-\frac{1}{\alpha}$ (d) $\frac{\alpha}{3}$

28. If α are α^2 are the roots of the equation $x^2 - 6x + c = 0$, then the positive value of c is

(a) 2 (b) 3 (c) 4 (d) 8

29. If one of the roots of the quadratic equation $ax^2 - bx + a = 0$ is 6 then the value of $\frac{b}{a}$ is equal to

(a) $\frac{1}{6}$ (b) $\frac{11}{6}$ (c) $\frac{37}{6}$ (d) $\frac{6}{11}$

30. If the equation $2x^2 + (a+3)x + 8 = 0$ has equal roots, then one of the values of a is

(a) –9 (b) –5 (c) –11 (d) 11

CATEGORY-II

Every correct answer will yield 2 marks. For incorrect response, 25% of full mark (1/2) would be deducted. If candidates mark more than one answer, negative marking will be done.

31. The area of the triangle formed by the line $x + y = 2$ and the angle bisectors of pair of straight lines $x^2 - y^2 + 2y = 1$ is a and coordinates of orthocentre of triangle are (b, c), then $4a + b + c$ is equal to

(a) 0 (b) 4 (c) 2 (d) 3

32. If $a\cos^2 3\alpha + b\cos^4\alpha = 16\cos^6\alpha + 9\cos^2\alpha$ is an identity, then

(a) $a = 1, b = 24$ (b) $a = 3, b = 24$
(c) $a = 4, b = 2$ (d) $a = 7, b = 18$

33. If α, γ are the roots of $x^2 + abx + c = 0$ and β, γ are the roots of $x^2 + acx + b = 0$ $(b \neq c)$, then the equation whose roots are α, β is

(a) $x^2 - a(b + c)x - a^2bc = 0$
(b) $x^2 - a(b - c)x + a^2bc = 0$
(c) $x^2 + a(b + c)x + a^2bc = 0$
(d) $x^2 + a(b + c)x - a^2bc = 0$

34. $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$

(a) is equal to zero
(b) lies between 0 and 3
(c) is a negative number
(d) lies between 3 and 6

35. A line passing through the point of intersection of $x + y = 4$ and $x - y = 2$ makes an angle $\tan^{-1}\frac{3}{4}$ with the x-axis. It intersects the parabola $y^2 = 4(x - 3)$ at points (x_1, y_1) and (x_2, y_2) respectively. Then $|x_1 - x_2| =$

(a) $\frac{16}{9}$ (b) $\frac{32}{9}$ (c) $\frac{40}{9}$ (d) $\frac{80}{9}$

CATEGORY-III

In this section more than 1 answer can be correct. Candidates will have to mark all the correct answers, for which 2 marks will be awarded. If, candidates marks one correct and one incorrect answer then no marks will be awarded. But if, candidate makes only correct, without making any incorrect, formula below will be used to allot marks. 2×(no. of correct response/total no. of correct options)

36. A line passes through the point $A(-5, -4)$ with slope $\tan\theta$ meets lines $x + 3y + 2 = 0$, $2x + y + 4 = 0$, $x - y - 5 = 0$ at B, C, D respectively, then which options are correct for the equation of the line.

(a) $\frac{15}{AB} = \cos\theta + 3\sin\theta$

(b) $\frac{10}{AC} = 2\cos\theta + \sin\theta$

(c) $\frac{6}{AD} = \cos\theta - \sin\theta$

(d) None of these

37. If the conjugate of $(x + iy)(1 - 2i)$ be $1 + i$, then

(a) $x = \frac{1}{5}$ (b) $x + iy = \frac{1}{5}(3+i)$

(c) $x - iy = \frac{1}{5}(3+i)$ (d) $x + iy = \frac{1-i}{1-2i}$

38. If $a + b + c = 0$, then the roots of the equation $(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$ can be

(a) imaginary (b) real and equal
(c) real and unequal (d) none of these

39. The value of $\left(\frac{\cos\alpha + \cos\beta}{\sin\alpha - \sin\beta}\right)^n + \left(\frac{\sin\alpha + \sin\beta}{\cos\alpha - \cos\beta}\right)^n$ (where n is a whole number) is equal to

(a) 0, when n is odd

(b) $2\tan^n\frac{\alpha-\beta}{2}, \forall n$

(c) $2\cot^n\frac{\alpha-\beta}{2}$, when n is even

(d) $2\cot^n\frac{\alpha+\beta}{2}$, when n is even

40. If $\tan^2 x + 2\tan x \tan 2y = \tan^2 y + 2\tan y \tan 2x = k$, then which of the following is correct?
(a) $k = 1$ (b) $\tan x = \tan y$
(c) $\tan x = -\tan y$ (d) 0

SOLUTIONS

1. (c): Given that $\arg\left(\frac{z-1}{z+1}\right)=\frac{\pi}{4} \Rightarrow \arg\frac{x+iy-1}{x+iy+1}=\frac{\pi}{4}$

$$\Rightarrow \arg\frac{\{(x-1+iy)(x+1-iy)\}}{(x+1)^2+y^2}=\frac{\pi}{4}$$

$$\Rightarrow \arg\frac{x^2+y^2-1+2yi}{(x+1)^2+y^2}=\frac{\pi}{4} \Rightarrow \frac{2y}{x^2+y^2-1}=1$$

$\Rightarrow x^2 + y^2 - 2y = 1$. This is the locus of a circle with centre (0,1).

2. (d) : $\frac{7+11\omega+13\omega^2}{13+7\omega+11\omega^2}+\frac{7+11\omega+13\omega^2}{11+13\omega+7\omega^2}$

$$=\frac{\omega(7+11\omega+13\omega^2)}{\omega(13+7\omega+11\omega^2)}+\frac{\omega(7+11\omega+13\omega^2)}{\omega(11+13\omega+7\omega^2)}$$

$$=\frac{1}{\omega}+\omega=\omega^2+\omega=-1$$

3. (c): Let $ABCD$ be the rhombus.
Coordinates of intersection of two sides of the rhombus, $A \equiv (1, 2)$
$\therefore$ Slope of $AE = 2 \Rightarrow$ Slope of $BD = -1/2$
(where E is the point of intersection of diagonals of rhombus)

$\Rightarrow$ Equation of BD is $\frac{y+2}{x+1}=-\frac{1}{2}$

$\Rightarrow x + 2y + 5 = 0$

$\therefore$ Coordinates of $D = \left(\frac{1}{3}, \frac{-8}{3}\right)$

4. (a) : $y=\frac{1}{2}\sin^2\theta+\frac{1}{3}\cos^2\theta=\frac{1}{2}-\frac{1}{6}\cos^2\theta$

$$=\frac{1}{2}-\frac{1}{12}(1+\cos 2\theta)=\frac{1}{2}-\frac{1}{12}-\frac{1}{12}\cos 2\theta$$

$$=\frac{5}{12}-\frac{1}{12}\cos 2\theta$$

$\therefore$ $y_{max}=\frac{5}{12}+\frac{1}{12}=\frac{1}{2}$ when $\cos 2\theta=-1$

and $y_{min}=\frac{5}{12}-\frac{1}{12}=\frac{1}{3}$ when $\cos 2\theta=1$

Thus $\frac{1}{3}\le y\le\frac{1}{2}$

5. (a) : Let the given vertices are $A(4, 0)$, $B(-1, -1)$, $C(3, 5)$
$AB^2 = (4 + 1)^2 + (0 + 1)^2 = 26$,
$BC^2 = (-1 - 3)^2 + (-1 - 5)^2 = 52$
and $AC^2 = (4 - 3)^2 + (0 - 5)^2 = 26$
$\because$ $AB^2 + AC^2 = 26 + 26 = 52 = BC^2$
$\therefore$ The triangle is isosceles and right-angled.

6. (a) : Distance between $x + y = 4$ and $2x + 2y = 5$ is
$$\frac{4}{\sqrt{2}}-\frac{5}{2\sqrt{2}}=\frac{3}{2\sqrt{2}}>1$$

7. (c): $\frac{x}{a}+\frac{y}{b}=1\Rightarrow bx+ay=ab$ ...(i)

Now, if (h, k) be the foot of the perpendicular drawn from the origin on (i), then $h=\frac{ab^2}{a^2+b^2}, k=\frac{a^2b}{a^2+b^2}$.

$$\therefore\ h^2+k^2=\frac{a^2b^4}{(a^2+b^2)^2}+\frac{a^4b^2}{(a^2+b^2)^2}=\frac{a^2b^2}{a^2+b^2}=c^2$$

$$\left(\because \frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}\right)$$

$\therefore$ Locus of the point (h, k) is $x^2 + y^2 = c^2$.

8. (a) : The equations of the sides of the triangle are $x-\sqrt{3}y=0, x+\sqrt{3}y=0$ and $x = 4$

$\therefore$ The vertices of the triangle are (0, 0), $\left(4, \frac{4}{\sqrt{3}}\right)$, $\left(4, -\frac{4}{\sqrt{3}}\right)$.

Thus, the triangle is isosceles.

9. (b) : $\sqrt[3]{x+iy}=a+ib$

$\Rightarrow$ $x+iy=(a+ib)^3=(a^3-3ab^2)+i(3a^2b-b^3)$
$\therefore$ $x = a^3 - 3ab^2$ and $y = 3a^2b - b^3$

Now, $\frac{x}{a}+\frac{y}{b}=\frac{a(a^2-3b^2)}{a}+\frac{b(3a^2-b^2)}{b}=4(a^2-b^2)$

10. (a) : Given that, $(1 + i)(1 + 2i)(1 + 3i)...(1 + ni) = a + ib$
Now $|(1+i)(1+2i)(1+3i)...(1+ni)|=|a+ib|$
$\Rightarrow$ $|1+i|^2\,|1+2i|^2\,|1+3i|^2 ...|1+ni|^2=|a+ib|^2$
$\Rightarrow$ $(1 + 1)(1 + 4)(1 + 9)...(1 + n^2) = a^2 + b^2$
$\Rightarrow$ $2 \times 5 \times 10 \times ... \times (1 + n^2) = a^2 + b^2$

11. (b) : The given equation is

$$(5+\sqrt{2})x^2-(4+\sqrt{5})x+8+2\sqrt{5}=0$$

Let the roots be a and b

$$\therefore\ a+b=\frac{4+\sqrt{5}}{5+\sqrt{2}} \text{ and } ab=\frac{8+2\sqrt{5}}{5+\sqrt{2}}$$

Now, the harmonic mean of the roots is

$$\frac{2ab}{a+b}=\frac{2(8+2\sqrt{5})}{4+\sqrt{5}}=4$$

12. (b) : Sum of the roots $=\sin 18°+\cos 36°$

$$=\frac{\sqrt{5}-1}{4}+\frac{\sqrt{5}+1}{4}=\frac{\sqrt{5}}{2}$$

Product of the roots = sin 18° × cos 36°

$$=\left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4}\right)=\frac{1}{4}$$

∴ The required equation is $x^2-\frac{\sqrt{5}}{2}x+\frac{1}{4}=0$

$\Rightarrow\ 4x^2-2\sqrt{5}x+1=0$

13. (a) : $\sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}+\sin^2\frac{5\pi}{8}+\sin^2\frac{7\pi}{8}$

$$=\sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}+\sin^2\left(\pi-\frac{3\pi}{8}\right)+\sin^2\left(\pi-\frac{\pi}{8}\right)$$

$$=\sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}+\sin^2\frac{3\pi}{8}+\sin^2\frac{\pi}{8}$$

$$=2\left(\sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}\right)=2\left(\sin^2\frac{\pi}{8}+\sin^2\left(\frac{\pi}{2}-\frac{\pi}{8}\right)\right)$$

$$=2\left(\sin^2\frac{\pi}{8}+\cos^2\frac{\pi}{8}\right)=2.$$

14. (a) : $\frac{\sqrt{3}}{\sin 15°}-\frac{1}{\cos 15°}=\frac{\sqrt{3}\cos 15°-\sin 15°}{\sin 15°\cos 15°}$

$$=\frac{\frac{\sqrt{3}}{2}\cos 15°-\frac{1}{2}\sin 15°}{\frac{1}{2}\sin 15°\cos 15°}=\frac{\cos 30°\cos 15°-\sin 30°\sin 15°}{\frac{1}{4}2\sin 15°\cos 15°}$$

$$=\frac{4\cos 45°}{\sin 30°}=4\sqrt{2}$$

15. (d) : Given that $\tan\theta=\frac{1}{2}$ and $\tan\phi=\frac{1}{3}$

We know $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$

$$\therefore\ \tan 2\theta=\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}=\frac{4}{3}$$

Now, $\tan(2\theta+\phi)=\frac{\tan 2\theta+\tan\phi}{1-\tan 2\theta\cdot\tan\phi}=\frac{\frac{4}{3}+\frac{1}{3}}{1-\frac{4}{3}\cdot\frac{1}{3}}=3$

16. (c): $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$

$$=\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\left(\pi-\frac{3\pi}{8}\right)\right)\left(1+\cos\left(\pi-\frac{\pi}{8}\right)\right)$$

$$=\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{\pi}{8}\right)$$

$$=\left(1-\cos^2\frac{\pi}{8}\right)\left(1-\cos^2\frac{3\pi}{8}\right)=\sin^2\frac{\pi}{8}\cdot\sin^2\frac{3\pi}{8}$$

$$=\frac{1}{4}\left(2\sin^2\frac{\pi}{8}\right)\left(2\sin^2\frac{3\pi}{8}\right)=\frac{1}{4}\left(1-\cos\frac{\pi}{4}\right)\left(1-\cos\frac{3\pi}{4}\right)$$

$$=\frac{1}{4}\left(1-\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{\sqrt{2}}\right)=\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{8}$$

17. (d) : The given equation of straight line is $4x-ay=8$

The intercept form of the straight line is $\frac{x}{2}+\frac{y}{\frac{-8}{a}}=1$

By the problem, we have

$$\frac{-8}{a}=3\cdot 2\ \Rightarrow\ a=-\frac{4}{3}$$

18. (a) : The equation of the straight line passing through the point (0, 1) is

$$y - 1 = m(x - 0) \Rightarrow mx - y + 1 = 0$$

(Where m is the slope of the line)

Now by the given condition, we have

$$\frac{1}{\sqrt{m^2+1}} = \frac{3}{5} \Rightarrow m = \pm\frac{4}{3}$$

So, the equation of lines are

$$\frac{4}{3}x - y + 1 = 0,\ -\frac{4}{3}x - y + 1 = 0$$

$\Rightarrow 4x - 3y + 3 = 0,\ 4x + 3y - 3 = 0$

19. (d) : Let $M(h, k)$ be any point on the line $4x + 3y = 25$

Then (slope of OM) $\times$ (slope of $4x + 3y = 25$) $= -1$

$$\Rightarrow \frac{k}{h} \times \frac{-4}{3} = -1 \Rightarrow k = \frac{3}{4}h \quad \text{... (1)}$$

Since $M(h, k)$ lies on $4x + 3y = 25$.

$\therefore\ 4h + 3k = 25$ ... (2)

On solving (1) and (2), we get $h = 4$, $k = 3$

Also, let $A(\alpha, \beta)$ be the image of the origin with respect to the line $4x + 3y = 25$

Then, $h = \dfrac{\alpha+0}{2}, k = \dfrac{\beta+0}{2} \Rightarrow \alpha = 8, \beta = 6$

So, the image of the origin $O(0, 0)$ is (8, 6).

20. (b) : $z = \dfrac{1+\sin\frac{\pi}{3}+i\cos\frac{\pi}{3}}{1+\sin\frac{\pi}{3}-i\cos\frac{\pi}{3}}$

$$= \frac{1+\frac{\sqrt{3}}{2}+\frac{i}{2}}{1+\frac{\sqrt{3}}{2}-\frac{i}{2}} = \frac{2+\sqrt{3}+i}{2+\sqrt{3}-i} \times \frac{2+\sqrt{3}+i}{2+\sqrt{3}+i}$$

$$= \frac{(3+2\sqrt{3})+i(2+\sqrt{3})}{4+2\sqrt{3}}$$

$$\therefore\ \arg(z) = \tan^{-1}\left(\frac{2+\sqrt{3}}{3+2\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

21. (d) : We have $\dfrac{(1+i)(2+3i)(3-4i)}{(2-3i)(1-i)(3+4i)} = a+ib$

$$\Rightarrow \left|\frac{(1+i)(2+3i)(3-4i)}{(2-3i)(1-i)(3+4i)}\right| = |a+ib|$$

$$\Rightarrow \frac{\sqrt{1+1}\times\sqrt{4+9}\times\sqrt{9+16}}{\sqrt{4+9}\times\sqrt{1+1}\times\sqrt{9+16}} = \sqrt{a^2+b^2}$$

$\Rightarrow a^2 + b^2 = 1$

22. (c): We have, $\arg(zw) = \pi$

$\Rightarrow \arg(z) + \arg(w) = \pi$ ... (i)

Also, $\overline{z+iw} = 0 \Rightarrow \bar{z} = i\bar{w} \Rightarrow \arg(\bar{z}) = \arg(i) + \arg(\bar{w})$

$$\Rightarrow \arg(z) - \arg(w) = -\frac{\pi}{2} \quad \text{...(ii)}$$

On solving (i) and (ii), we get $\arg(z) = \dfrac{\pi}{4}$

23. (a) : We have $z = \dfrac{2-i}{i} \Rightarrow z^2 = \dfrac{(2-i)^2}{i^2} = -3+4i$

$\therefore\ \text{Re}(z^2) + \text{Im}(z^2) = -3 + 4 = 1$

24. (c): Let $z = x + iy$

Then, $|z + 1| < |z - 1| \Rightarrow |x + iy + 1| < |x + iy - 1|$

$$\Rightarrow \sqrt{(x+1)^2+y^2} < \sqrt{(x-1)^2+y^2}$$

$\Rightarrow x^2 + 1 + 2x < x^2 + 1 - 2x$

$\Rightarrow 4x < 0 \Rightarrow x < 0$

25. (c): We have, $\left|z - \dfrac{3}{z}\right| = 2$

$$\text{Now, } |z| = \left|\left(z-\frac{3}{z}\right)+\frac{3}{z}\right| \le \left|z-\frac{3}{z}\right| + \left|\frac{3}{z}\right| = 2 + \left|\frac{3}{z}\right|$$

$$\Rightarrow |z| - \frac{3}{|z|} \le 2 \Rightarrow |z|^2 - 2|z| - 3 \le 0$$

$\Rightarrow (|z| + 1)(|z| - 3) \le 0$

$\Rightarrow |z| \ge -1$ or $|z| \le 3$

So, the greatest value of $|z|$ is 3.

26. (d) : Given that tan33° and tan12° are the roots of the equation $mx^2 - nx + k = 0$

$$\therefore\ \tan 33° + \tan 12° = \frac{n}{m} \text{ and } \tan 33° \tan 12° = \frac{k}{m}$$

$$\text{Now, } \frac{2m+n+k}{m} = 2 + \frac{n}{m} + \frac{k}{m}$$

$= 2 + (\tan 33° + \tan 12°) + (\tan 33° \times \tan 12°)$

$= 2 + 1 = 3$ [$\because \tan(33° + 12°) = \tan 45° = 1$]

27. (a) : According to question,

$$\alpha\beta = \frac{-1}{4} \Rightarrow \beta = -\frac{1}{4\alpha}$$

28. (d) : Given that α and α^2 are the roots of

$x^2 - 6x + c = 0$

$\therefore\ \alpha + \alpha^2 = 6$ and $\alpha^3 = c$

$\Rightarrow \alpha^2 + \alpha - 6 = 0 \Rightarrow (\alpha+3)(\alpha-2) = 0$

$\Rightarrow \alpha = -3$ or $\alpha = 2$

$\therefore\ c = -27$ or 8 [$\because \alpha^3 = c$]

29. (c): Let the other root be β

Then, $6\beta = 1 \Rightarrow \beta = \frac{1}{6}$

Also, $6 + \beta = \frac{b}{a} \Rightarrow \frac{b}{a} = 6 + \frac{1}{6} = \frac{37}{6}$

30. (c): Given that the roots of the equation $2x^2 + (a + 3)x + 8 = 0$ are equal

$\therefore \quad (a + 3)^2 = 4 \times 2 \times 8 = 64$

$\Rightarrow \quad a + 3 = \pm 8 \quad \Rightarrow \quad a = 5 \text{ or } a = -11$

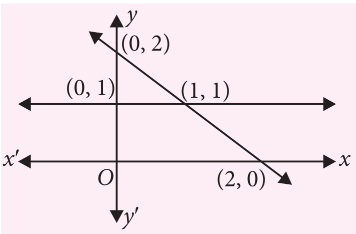
31. (d) : The pair of straight lines, $x^2 - y^2 + 2y - 1 = 0$

$\Rightarrow \quad x^2 - (y^2 - 2y + 1) = 0$

$\Rightarrow \quad (x + y - 1)(x - y + 1) = 0$

Angle bisector of given pair of lines is

$$\frac{x + y - 1}{\sqrt{2}} = \pm \frac{x - y + 1}{\sqrt{2}}$$



Taking positive sign on right hand side, we get

$x + y - 1 = x - y + 1 \Rightarrow 2(y - 1) = 0 \Rightarrow y = 1$

Taking negative sign on right hand side, we get

$x + y - 1 = -x + y - 1 \Rightarrow 2x = 0 \Rightarrow x = 0$

Area $= a = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

Orthocentre $(0, 1) \Rightarrow b = 0, c = 1$

Then $4a + b + c = 2 + 0 + 1 = 3$

32. (a) : $16\cos^6\alpha + 9\cos^2\alpha = (4\cos^3\alpha)^2 + 9\cos^2\alpha$

$= (\cos 3\alpha + 3\cos\alpha)^2 + 9\cos^2\alpha$

$= \cos^2 3\alpha + 18\cos^2\alpha + 6\cos 3\alpha \cdot \cos\alpha$

$= \cos^2 3\alpha + 18\cos^2\alpha + 3(\cos 4\alpha + \cos 2\alpha)$

$= \cos^2 3\alpha + 18\cos^2\alpha + 3\cos 4\alpha + 3(2\cos^2\alpha - 1)$

$= \cos^2 3\alpha + 24\cos^2\alpha + 3(\cos 4\alpha - 1)$

$= \cos^2 3\alpha + 24\cos^2\alpha - 6\sin^2 2\alpha$

$= \cos^2 3\alpha + 24\cos^2\alpha - 24\sin^2\alpha\cos^2\alpha$

$= \cos^2 3\alpha + 24\cos^2\alpha(1 - \sin^2\alpha)$

$= \cos^2 3\alpha + 24\cos^4\alpha$

$16\cos^6\alpha + 9\cos^2\alpha = a\cos^2 3\alpha + b\cos^4\alpha$ (Given)

Comparing, we get $a = 1, b = 24$

33. (a) : α and γ are the roots of the equation $x^2 + abx + c = 0$

$\therefore \quad \alpha + \gamma = -ab, \alpha\gamma = c$

β and γ are the roots of the equation $x^2 + acx + b = 0$

$\therefore \quad \beta + \gamma = -ac, \beta\gamma = b$

Now $\alpha - \beta = a(c - b)$ and $(\alpha - \beta)\gamma = c - b$

$\therefore \quad a = \frac{1}{\gamma}$

Hence, $\alpha = ac$ and $\beta = ab \therefore \alpha + \beta = a(b + c)$ and $\alpha\beta = a^2bc$

$\therefore$ The required equation is $x^2 - a(b + c)x + a^2bc = 0$

34. (c): $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$

$= \cos\theta + \cos 2\theta + \cos 3\theta \quad [\text{Let } \frac{2\pi}{7} = \theta]$

$= \frac{1}{2\sin\frac{\theta}{2}}\left(2\sin\frac{\theta}{2}\cos\theta + 2\sin\frac{\theta}{2}\cos 2\theta + 2\sin\frac{\theta}{2}\cos 4\theta\right)$

$[\therefore \cos 3\theta = \cos(7\theta - 4\theta) = \cos(2\pi - 4\theta) = \cos 4\theta]$

$= \frac{1}{2\sin\frac{\theta}{2}}\left(\sin\frac{3\theta}{2} - \sin\frac{\theta}{2} + \sin\frac{5\theta}{2} - \sin\frac{3\theta}{2} + \sin\frac{9\theta}{2} - \sin\frac{7\theta}{2}\right)$

$= \frac{1}{2\sin\frac{\theta}{2}}\left(-\sin\frac{\theta}{2} + \sin\frac{9\theta}{2} + \sin\frac{5\theta}{2} - \sin\pi\right) \quad (\because \frac{2\pi}{7} = \theta)$

$= \frac{1}{2\sin\frac{\theta}{2}}\left(-\sin\frac{\theta}{2} + 2\sin\frac{7\theta}{2}\cos\theta\right)$

$= \frac{1}{2\sin\frac{\theta}{2}}\left(-\sin\frac{\theta}{2}\right) \quad (\because \sin\frac{7\theta}{2} = \sin\pi = 0)$

$= -\frac{1}{2}$

35. (b) : Given that the line is passing through the point (3, 1) (the point of intersection of the straight lines $x + y = 4$ and $x - y = 2$) and its slope is 3/4.

$\therefore$ The equation of the line is

$$y - 1 = \frac{3}{4}(x - 3) \Rightarrow 3x - 4y = 5 \quad \text{...(1)}$$

Thus the straight line (1) cuts the parabola $y^2 = 4(x - 3)$ at (x_1, y_1) and (x_2, y_2)

Now putting $y = \frac{3x - 5}{4}$ in the equation of the parabola, we get

$$\left(\frac{3x - 5}{4}\right)^2 = 4(x - 3) \Rightarrow 9x^2 - 94x + 217 = 0$$

Let x_1 and x_2 are the roots of the equation.

$\therefore \quad x_1+x_2=\frac{94}{9}$ and $x_1x_2=\frac{217}{9}$

$\therefore \quad (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$

$\therefore \quad (x_1-x_2)^2=\left(\frac{94}{9}\right)^2-4\times\frac{217}{9}=\frac{1024}{81}$

$\therefore \quad |x_1-x_2|=\frac{32}{9}$

36. (a, b, c) : Equation of the line passing through $A(-5, -4)$ with slope $\tan\theta$ is $y + 4 = \tan\theta(x + 5)$

$\Rightarrow \quad (\tan\theta)x - y + 5\tan\theta - 4 = 0 \qquad \text{... (1)}$

Now, solving (1) and $x + 3y + 2 = 0$ we get the coordinates $B\left(\frac{10-15\tan\theta}{3\tan\theta+1}, \frac{3\tan\theta-4}{3\tan\theta+1}\right)$

Now,

$$AB=\sqrt{\left(-5-\frac{10-15\tan\theta}{3\tan\theta+1}\right)^2+\left(-4-\frac{3\tan\theta-4}{3\tan\theta+1}\right)^2}$$

$$=\sqrt{\frac{15^2+(15\tan\theta)^2}{(3\tan\theta+1)^2}}=\frac{15\sec\theta}{3\tan\theta+1}=\frac{15}{3\sin\theta+\cos\theta}$$

$$\therefore \quad \frac{15}{AB}=3\sin\theta+\cos\theta$$

Solving (1) and $2x + y + 4 = 0$, we get the coordinate $C\left(\frac{-5\tan\theta}{2+\tan\theta}, \frac{6\tan\theta-8}{2+\tan\theta}\right)$

Now,

$$AC=\sqrt{\left(-5+\frac{5\tan\theta}{2+\tan\theta}\right)^2+\left(-4-\frac{6\tan\theta-8}{2+\tan\theta}\right)^2}$$

$$=\sqrt{\frac{10^2+(10\tan\theta)^2}{(2+\tan\theta)^2}}=\frac{10\sec\theta}{2+\tan\theta}=\frac{10}{2\cos\theta+\sin\theta}$$

$$\therefore \quad \frac{10}{AC}=2\cos\theta+\sin\theta$$

Solving (1) and $x - y - 5 = 0$, we get the coordinate of

$$D\left(\frac{5\tan\theta+1}{1-\tan\theta}, \frac{4-10\tan\theta}{\tan\theta - 1}\right)$$

$$\therefore \quad \frac{6}{AD}=\sin\theta-\cos\theta$$

37. (b, d) : Here $(x+iy)(1-2i)=x+iy-2xi+2y$
$=(x+2y)+(y-2x)i$

According to question, $(x + 2y) - (y - 2x)i = 1 + i$

$\therefore \quad x + 2y = 1$ and $2x - y = 1$

Solving the above equations, we get

$$x=\frac{3}{5}, y=\frac{1}{5} \quad \therefore x+iy=\frac{1}{5}(3+i)$$

and $\frac{1-i}{1-2i}=\frac{(1-i)(1+2i)}{1^2-4i^2}=\frac{1}{5}(3+i)=x+iy$

38. (b, c): Let $A = b + c - a$, $B = c + a - b$ and $C = a + b - c$.

Now, $A + B + C = a + b + c$

$\Rightarrow \quad A + B + C = 0$

Now, the given equation is

$(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$

$\Rightarrow \quad Ax^2 + Bx + C = 0$

Now, $B^2 - 4AC = (A + C)^2 - 4AC$

$= (A - C)^2 = (b + c - a - a - b + c)^2 = 4(c - a)^2$

i.e. $B^2 - 4AC$ is a perfect square.

39. (a, c): Here $\left(\frac{\cos\alpha+\cos\beta}{\sin\alpha-\sin\beta}\right)^n+\left(\frac{\sin\alpha+\sin\beta}{\cos\alpha-\cos\beta}\right)^n$

$$=\left(\frac{2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}\right)^n+\left(\frac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}}\right)^n$$

$$=\left(\cot\frac{\alpha-\beta}{2}\right)^n+\left(-\cot\frac{\alpha-\beta}{2}\right)^n$$

$$=\begin{cases} 0 & \text{, when } n \text{ is odd} \\ 2\cot^n\frac{\alpha-\beta}{2} & \text{, when } n \text{ is even}\end{cases}$$

40. (a, b, c) : Let

$$\tan^2 x+2\tan x\tan 2y=\tan^2 y+2\tan y\tan 2x=k$$

From $\tan^2x + 2\tan x \tan 2y = k$, we have

$$\tan 2y=\frac{k-\tan^2 x}{2\tan x}\Rightarrow\frac{2\tan y}{1-\tan^2 y}=\frac{k-\tan^2 x}{2\tan x}$$

$\Rightarrow \quad (k - \tan^2x)(1 - \tan^2y) = 4\tan x \tan y$

Similarly from the other relation,

$\Rightarrow \quad (k - \tan^2y)(1 - \tan^2x) = 4\tan x \tan y$

Thus, $(k - \tan^2x)(1 - \tan^2y) = (k - \tan^2y)(1 - \tan^2x)$

Simplifying, we get

$(k - 1)(\tan^2x - \tan^2y) = 0$

Thus, either $k = 1$ or $\tan x = \pm\tan y$

MPP-3 CLASS XII ANSWER KEY

1.	(c)	**2.**	(d)	**3.**	(c)	**4.**	(a)	**5.**	(b)
6.	(d)	**7.**	(a,b,c)	**8.**	(a,b)	**9.**	(a,b,c,d)		
10.	(a,b,c,d)	**11.**	(a,b)	**12.**	(a,b)	**13.**	(a,b,c)	**14.**	(b)
15.	(c)	**16.**	(c)	**17.**	(6)	**18.**	(0)	**19.**	(2)
20.	(1)								

OLYMPIAD CORNER

1. Determine the number of non-negative integers x that satisfy the equation $\left[\frac{x}{44}\right]=\left[\frac{x}{45}\right]$.

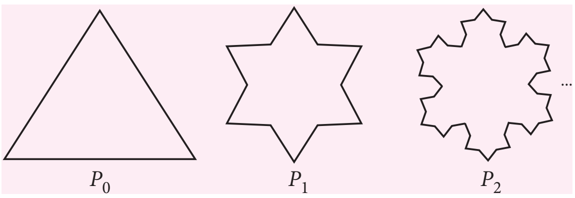
(Note : If r is any real number, then $[r]$ denotes the greatest integer less than or equal to r).

2. As shown in the diagram, there is a sequence of the curves $P_0, P_1, P_2, \ldots$. It is known that the region enclosed by P_0 has area 1 and P_0 is an equilateral triangle. We obtain P_{k+1} from P_k by operating as follows :

Trisecting every side of P_k, then we construct an equilateral triangle outwardly on every side of P_k sitting on the middle segment of the side and finally remove this middle segment ($k = 0, 1, 2, \ldots$). Write S_n as the area of the region enclosed by P_n.

(1) Find a formula for the general term of the sequence of numbers $\{S_n\}$

(2) Find $\lim\limits_{n\to\infty} S_n$.



3. A circle with centre O and radius R is drawn on a paper, and A is a given point in the circle with $OA = a$. Fold the paper to make a point A' on the circumference coincident with point A, then a crease line is left on the paper. Find out the set of all points on such crease lines, when A' goes through every point on the circumference.

4. The rule of an "obstacle course" specifies that at the n^{th} obstacle a person has to toss a die n times. If the sum of points in these n tosses is bigger than 2^n, the person is said to have crossed the obstacle.

(1) At most how many obstacles can a person cross?

(2) What is the probability that a person crosses the first three obstacles?

(**Note :** A die is a fair regular cube, on its six faces there are numbers 1, 2, 3, 4, 5, 6 respectively. Toss a die, the point is the number appearing on its top face after it stops moving.)

5. Determine the smallest positive integer y for which there is a positive integer x satisfying the equation $2^{13} + 2^{10} + 2^x = y^2$.

SOLUTIONS

1. 1st solution :

$$\text{Let } \left[\frac{x}{44}\right]=\left[\frac{x}{45}\right]=n.$$

Since x is non-negative, n is also non-negative.

If $n = 0$, then x is any integer from 0 to $44 - 1 = 43$: a total of 44 values.

If $n = 1$, then x is any integer from 45 to $2 \times 44 - 1 = 87$: a total of 43 values.

If $n = 2$, then x is any integer from $2 \times 45 = 90$ to $3 \times 44 + 1 = 131$: a total of 42 values.

If $n = k$, then x is any integer from $45k$ to $44 (k - 1) - 1 = 44k + 43$: a total of $(44k + 43) - (45k - 1) = 44 - k$ values.

Thus, increasing n by 1 decreases the number of values of x by 1. Also the largest value of n is 43, in which case x has only 1 value.

Therefore the number of non-negative integer values of x is $44+43+\ldots+1=\frac{1}{2}(44\times 45)=990.$

2^nd^ solution :

Let n be a non-negative integer such that

$$\left[\frac{x}{44}\right]=\left[\frac{x}{45}\right]=n$$

Then $\left[\frac{x}{44}\right]=n \Leftrightarrow 44n\le x<44(n+1)$ and

$$\left[\frac{x}{45}\right]=n \Leftrightarrow 45n\le x<45(n+1)$$

So, $\left[\frac{x}{44}\right]=\left[\frac{x}{45}\right]=n$

$\Leftrightarrow 45n \le x < 44(n+1)$

$\Leftrightarrow 44n + n \le x < 44n + 44$

This is the case if and only if $n < 44$ and then x can assume exactly $44 - n$ different values.

Therefore the number of non-negative integer values of x is

$(44 - 0) + (44 - 1) + ... + (44 - 43)$

$$= 44 + 43 + ... + 1 = \frac{1}{2}(44\times 45)=990$$

3^rd^ solution :

Let n be a non-negative integer such that

$$\left[\frac{x}{44}\right]=\left[\frac{x}{45}\right]=n$$

Then $x = 44n + r$ where $0 \le r \le 43$ and

$x = 45n + s$, where $0 \le s \le 44$

So $n = r - s$. Therefore $0 \le n \le 43$.

Also $r = n + s$. Therefore $n \le r \le 43$

Therefore the number of non-negative integer values of x is $44 + 43 + ... + 1 = \frac{1}{2}(44\times 45)=990$

2. (1) We perform the operation on P_0. It is easy to see that each side of P_0 becomes 4 sides of P_1. So the number of sides of P_1 is $3\cdot 4$. In the same way, we operate on P_1. Each sides of P_1 becomes 4 sides of P_2. So the number of sides of P_2 is $3\cdot 4^2$. Consequently, it is not difficult to get that the number of sides of P_n is $3\cdot 4^n$.

It is known that the area of P_0 is $S_0 = 1$. Comparing P_1 with P_0, it is easy to see that we add to P_1 a smaller equilateral triangle with area $\frac{1}{3^2}$ on each side of P_0. Since P_0 has 3 sides, so $S_1 = S_0 + 3\cdot\frac{1}{3^2} = 1 + \frac{1}{3}$.

Again, comparing P_2 with P_1, we see that P_2 has an additional smaller equilateral triangle with area $\frac{1}{3^2}\cdot\frac{1}{3^2}$ on each side of P_1 and P_1 has $3\cdot 4$ sides. So that

$$S_2=S_1+3\cdot 4\cdot\frac{1}{3^4}=1+\frac{1}{3}+\frac{4}{3^3}$$

Similarly, we have

$$S_3=S_2+3\cdot 4\cdot\frac{1}{3^6}=1+\frac{1}{3}+\frac{4}{3^3}+\frac{4^2}{3^5}.$$

Hence, we have

$$S_n=1+\frac{1}{3}+\frac{4}{3^3}+\frac{4^2}{3^5}+\ldots+\frac{4^{n-1}}{3^{2n-1}}$$

$$=1+\sum_{k=1}^{n}\frac{4^{k-1}}{3^{2k-1}}=1+\frac{3}{4}\sum_{k=1}^{n}\left(\frac{4}{9}\right)^k$$

$$=1+\frac{3}{4}\cdot\frac{\frac{4}{9}\cdot\left[1-\left(\frac{4}{9}\right)^n\right]}{1-\frac{4}{9}}=1+\frac{3}{5}\left[1-\left(\frac{4}{9}\right)^n\right]$$

$$=\frac{8}{5}-\frac{3}{5}\cdot\left(\frac{4}{9}\right)^n \qquad (*)$$

We will prove (*) by mathematical induction as follows :

When $n = 1$, it is known that (*) holds from above.

Suppose, when $n = k$, we have

$$S_k=\frac{8}{5}-\frac{3}{5}\cdot\left(\frac{4}{9}\right)^k.$$

When $n = k + 1$, it is easy to see that, after $k + 1$ times of operations, by comparing P_{k+1} with P_k, we have added to P_{k+1} a smaller equilateral triangle with area $\frac{1}{3^{2(k+1)}}$ on each side of P_k and P_k has $3\cdot 4^k$ sides. So we get

$$S_{k+1}=S_k+3\cdot 4^k\cdot\frac{1}{3^{2(k+1)}}$$

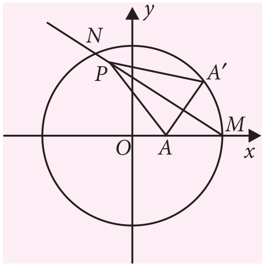
$$=S_k+\frac{4^k}{3^{2k+1}}=\frac{8}{5}-\frac{3}{5}\cdot\left(\frac{4}{9}\right)^{k+1}.$$

By mathematical induction, (*) is proved.

(2) From (1), we have $S_n=\frac{8}{5}-\frac{3}{5}\cdot\left(\frac{4}{9}\right)^n$.

Therefore, $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{8}{5}-\frac{3}{5}\cdot\left(\frac{4}{9}\right)^n\right]=\frac{8}{5}.$

3. Establish an xy-coordinate system as in the diagram with $A(a, 0)$ given. Then the crease line MN is the perpendicular bisector of segment AA' when $A'(R \cos a, R \sin a)$ is made coincident with A by folding the paper.



Let $P(x, y)$ be any point on MN, then $|PA'| = |PA|$.
i.e., $(x - R\cos a)^2 + (y - R\sin a)^2 = (x - a)^2 + y^2$.

Then, $\dfrac{x\cos a + y\sin a}{\sqrt{x^2+y^2}} = \dfrac{R^2-a^2+2ax}{2R\sqrt{x^2+y^2}}$.

We get $\sin(\theta+a)=\dfrac{R^2-a^2+2ax}{2R\sqrt{x^2+y^2}}$,

where $\sin\theta=\dfrac{x}{\sqrt{x^2+y^2}}, \cos\theta=\dfrac{y}{\sqrt{x^2+y^2}}$.

So, $\left|\dfrac{R^2-a^2+2ax}{2R\sqrt{x^2+y^2}}\right|\le 1$

Squaring both sides, we get

$$\frac{(2x-a)^2}{R^2}+\frac{4y^2}{R^2-a^2}\ge 1$$

So the set we want consists of all of the points on the border of or outside the ellipse

$$\frac{(2x-a)^2}{R^2}+\frac{4y^2}{R^2-a^2}=1$$

Remark : As seen in the diagram, suppose the crease line intersects OA' at point Q. Then from $QA = QA'$ we have $OQ + QA = QA' = R$; that is, Q is on an ellipse whose foci are O and A. And the expression of the ellipse is

$$\frac{\left(x-\frac{a}{2}\right)^2}{\left(\frac{R}{2}\right)^2}+\frac{y^2}{\left(\frac{1}{2}\sqrt{R^2-a^2}\right)^2}=1$$

For any other point P on the crease line MN, we always have $PO + PA = PO + PA' > OA' = R$. So a point on the crease line is either on the border of or outside the ellipse. On the other hand, if from any point P on the border or outside the ellipse, we draw a line tangent to the ellipse at point Q, then $QO + QA = R$. Suppose line QO intersects the circle at point A'. Then $QO + QA' = R$, that is, $QA = QA'$. Then using the property of the tangent to an ellipse, we have that PQ bisects $\angle AQA'$, and that means the tangent PQ is the crease line as mentioned above. The arguments above reveal that the set we find consists of all of the points on every tangent to the ellipse.

4. Since the die is fair, the probability of any of the six numbers appearing is the same.
(1) Since the highest point of a die is 6 and $6 \times 4 > 2^4$, $6 \times 5 < 2^5$, it is impossible that the sum of points appearing in n tosses is bigger than 2^n if $n \ge 5$. This means it is an impossible event and the probability of crossing the obstacle is 0.
Therefore at most 4 obstacles that a person can cross.
(2) We denote A_n the event "at the n^{th} obstacle the person fails to cross", the complementary event $\overline{A_n}$ is "at the n^{th} obstacle the person crosses successfully".
At the n^{th} obstacle of this game the number of all possible outcomes is 6^n.
The first obstacle : event A_1 contains 2 possible outcomes (*i.e.*, the outcomes in which the number appearing is 1 or 2). So the probability of crossing the obstacle is

$$P(\overline{A_1})=1-P(A_1)=1-\frac{2}{6}=\frac{2}{3}.$$

The second obstacle : the number of outcomes contained in event A_2 is the total number of positive integer solution sets of the equation $x + y = a$, where a is taken to be 2, 3 and 4 respectively. Thus the number of outcomes equals ${}^1C_1 + {}^2C_1 + {}^3C_1 = 1 + 2 + 3 = 6$ and the probability of crossing the obstacle is

$$P(\overline{A_2})=1-P(A_2)=1-\frac{6}{6^2}=\frac{5}{6}.$$

The third obstacle : the number of outcomes contained in event A_3 is the total number of positive integer solution sets of the equation $x + y + z = a$, where a is taken to be 3, 4, 5, 6, 7, and 8 respectively. Thus the number of outcomes equal
${}^2C_2 + {}^3C_2 + {}^4C_2 + {}^5C_2 + {}^6C_2 + {}^7C_2$
$= 1 + 3 + 6 + 10 + 15 + 21 = 56$
and the probability of crossing the obstacle is

$P(\overline{A_3}) = 1 - P(A_3) = 1 - \frac{56}{6^3} = \frac{20}{27}.$

Consequently, the probability that a person crosses the first three obstacles is

$P(\overline{A_1}) \times P(\overline{A_2}) \times P(\overline{A_3}) = \frac{2}{3} \times \frac{5}{6} \times \frac{20}{27} = \frac{100}{243}.$

(We can also list all the possible outcomes at the second obstacle and at the third obstacle.)

Remark : Problems concerning probability theory first appeared in the National High School Mathematics Competition. Problems are not too difficult. Topics such as derivative and its applications have already appeared in high school textbooks. These topics will also appear in mathematics competitions.

5. 1st solution :

$2^{13} + 2^{10} + 2^x = y^2 \Leftrightarrow 2^{10}(2^3 + 1) + 2^x = y^2$

$\Leftrightarrow (2^5 \times 3)^2 + 2^x = y^2 \Leftrightarrow 2^x = y^2 - 96^2$

$\Leftrightarrow 2^x = (y + 96)(y - 96)$

Since y is an integer, both $y + 96$ and $y - 96$ must be powers of 2.

Let $y + 96 = 2^m$ and $y - 96 = 2^n$.

Then $2^m - 2^n = 192 = 2^6 \times 3$.

Hence, $2^{m-6} - 2^{n-6} - = 3$.

So, $2^{m-6} = 4$ and $2^{n-6} = 1$.

In particular, $m = 8$.

Hence, $y = 2^8 - 96 = 256 - 96 = 160$.

2nd solution :

We have $y^2 = 2^{13} + 2^{10} + 2^x = 2^{10}(2^3 + 1 + 2^{x-10})$
$= 2^{10}(9 + 2^{x-10})$

So we want the smallest value of $9 + 2^{x-10}$ that is a perfect square.

Since $9 + 2^{x-10}$ is odd and greater than 9, $9 + 2^{x-10} \geq 25$

Since $9 + 2^{41-10} = 25$, $y = 2^5 \times 5 = 32 \times 5 = 160$.

Comment

1st solution show that $2^{13} + 2^{10} + 2^x = y^2$ has only one solution.

YOU ASK WE ANSWER

Do you have a question that you just can't get answered?

Use the vast expertise of our MTG team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough.

The best questions and their solutions will be printed in this column each month.

1. Show that the difference of the squares of the tangents to two coplanar circles from any point P in the plane of the circles varies as the perpendicular from P on their radical axis. Also prove that the locus of a point such that the difference of the squares of the tangents from it to two given circles is constant, is a line parallel to their radical axis.

Mukul Kumar, H.P.

Ans. Let the two circles be

$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ ...(i)

and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ ...(ii)

and let $P = (h, k)$

$\therefore$ Radical axis of (i) and (ii) is

$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$ ...(iii)

Let length of tangents from $P(h, k)$ on (i) and (ii) are l_1 and l_2 then

$$l_1 = \sqrt{S_1} = \sqrt{(h^2 + k^2 + 2g_1h + 2f_1k + c_1)}$$

and $$l_2 = \sqrt{S_2} = \sqrt{(h^2 + k^2 + 2g_2h + 2f_2k + c_2)}$$

Consider,

$$l_1^2 - l_2^2 = 2(g_1 - g_2)h + 2(f_1 - f_2)k + c_1 - c_2 \quad \text{...(iv)}$$

Let p be the perpendicular distance from $P(h, k)$ on (iii),

$$\therefore \quad p = \frac{|2(g_1 - g_2)h + 2(f_1 - f_2)k + c_1 - c_2|}{\sqrt{4(g_1 - g_2)^2 + 4(f_1 - f_2)^2}} \quad \text{...(v)}$$

From (iv) and (v), we get

$$p = \frac{|l_1^2 - l_2^2|}{2\sqrt{(g_1 - g_2)^2 + (f_1 - f_2)^2}}$$

or $$\frac{|l_1^2 - l_2^2|}{p} = 2\sqrt{(g_1 - g_2)^2 + (f_1 - f_2)^2} = \text{constant}$$

$$\therefore \quad |l_1^2 - l_2^2| \propto p$$

Locus of $P(h, k)$ in (iv), is

$$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = (l_1^2 - l_2^2)$$

which is a line parallel to (iii).

2. A piece of wire of length $4L$ is bent at random, to form a rectangle. Then what is the probability that its area is at most $\frac{L^2}{4}$.

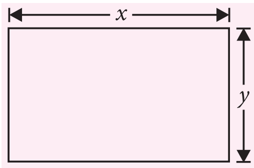
Kanchan, New Delhi

Ans. According to question, $2x + 2y = 4L$

$\Rightarrow \quad x + y = 2L$

$\therefore$ Area of the rectangle $= xy = x(2L - x)$

But given, $x(2L - x) \le \frac{L^2}{4}$



$$\Rightarrow \quad x^2 - 2Lx + \frac{L^2}{4} \ge 0$$

$$\Rightarrow \quad (x - L)^2 - L^2 + \frac{L^2}{4} \ge 0$$

$$\Rightarrow \quad (x - L)^2 - \left(\frac{L\sqrt{3}}{2}\right)^2 \ge 0$$

$$\Rightarrow \quad \left(x - L + \frac{L\sqrt{3}}{2}\right)\left(x - L - \frac{L\sqrt{3}}{2}\right) \ge 0$$

$$\Rightarrow \quad \left\{x - \left(\frac{2 - \sqrt{3}}{2}\right)L\right\}\left\{x - \left(\frac{2 + \sqrt{3}}{2}\right)L\right\} \ge 0$$

$$\therefore \quad x \in \left(0, \left(\frac{2 - \sqrt{3}}{2}\right)L\right] \cup \left[\left(\frac{2 + \sqrt{3}}{2}\right)L, 2L\right)$$

$\therefore$ Required probability

$$= \frac{\int_0^{\left(\frac{2-\sqrt{3}}{2}\right)L} dx + \int_{\left(\frac{2+\sqrt{3}}{2}\right)L}^{2L} dx}{\int_0^{2L} dx}$$

$$= \frac{\left(\frac{2-\sqrt{3}}{2}\right)L + 2L - \left(\frac{2+\sqrt{3}}{2}\right)L}{2L - 0}$$

$$= \frac{2L - \sqrt{3}L}{2L} = \left(\frac{2-\sqrt{3}}{2}\right)$$

3. Let N denotes the greatest number of points in which m straight lines and n circles intersect, then prove that $N - {}^mC_2 - {}^nP_2$ is an even integer.

Mahesh Jain, W.B.

Ans. m straight lines can intersect in at most ${}^mC_2$ points and n circles can intersect in at most $2 \times {}^nC_2 = n(n-1)$ $= {}^nP_2$ points.

n straight lines and m circles can intersect in at most $2 \times m \times n = 2\ mn$ points

Hence, $N = {}^mC_2 + {}^nP_2 + 2\ mn$

or $(N - {}^mC_2 - {}^nP_2) = 2\ mn$ which is an even integer

4. Integrate $\frac{\sqrt{\cos(2x)}}{\sin x}$ with respect to x

Shubham Verma, U.P.

Ans. Let $I = \int \frac{\sqrt{\cos 2x}}{\sin x} dx$

$$\Rightarrow \quad I = \int \frac{\cos 2x}{\sin x\sqrt{\cos 2x}} dx$$

$$\Rightarrow \quad I = \int \frac{1 - 2\sin^2 x}{\sin x\sqrt{\cos 2x}} dx$$

$$\Rightarrow \quad I = \int \frac{1}{\sin x\sqrt{\cos 2x}} dx - \int \frac{2\sin x}{\sqrt{\cos 2x}} dx$$

$$\Rightarrow \quad I = I_1 - I_2 \qquad \text{...(i)}$$

Now, $I_1 = \int \frac{1}{\sin x\sqrt{\cos 2x}} dx$

$$= \int \frac{1}{\sin x\sqrt{\cos^2 x - \sin^2 x}} dx = \int \frac{1}{\sin^2 x\sqrt{\cot^2 x - 1}} dx$$

$$\Rightarrow \quad I_1 = \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cot^2 x - 1}} dx$$

Now, put $\cot x = t \Rightarrow -\operatorname{cosec}^2 x\, dx = dt$

$$\therefore \quad I_1 = -\int \frac{dt}{\sqrt{t^2 - 1}}$$

$$\Rightarrow \quad I_1 = -\log\left|t + \sqrt{t^2 - 1}\right| + c_1$$

$$\Rightarrow \quad I_1 = -\log\left|\cot x + \sqrt{\cot^2 x - 1}\right| + c_1 \qquad \text{...(ii)}$$

And $I_2 = \int \frac{2\sin x}{\sqrt{\cos 2x}} dx$

$$= \int \frac{2\sin x}{\sqrt{2\cos^2 x - 1}} dx$$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$

$$\therefore \quad I_2 = -2\int \frac{dt}{\sqrt{2t^2 - 1}} = -\sqrt{2}\int \frac{dt}{\sqrt{t^2 - \left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$= -\sqrt{2}\log\left|t + \sqrt{t^2 - 1/2}\right| + c_2$$

$$= -\sqrt{2}\log\left|\cos x + \sqrt{\frac{2\cos^2 x - 1}{2}}\right| + c_2 \qquad \text{...(iii)}$$

From (i), (ii) and (iii), we get

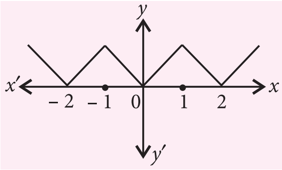
$$I = -\log\left|\cot x + \sqrt{\cot^2 x - 1}\right| + \sqrt{2}\log\left|\cos x + \sqrt{\frac{\cos 2x}{2}}\right| + c$$

Where c is the constant of integration.

MATHS MUSING

SOLUTION SET-164

1. **(d) :**



$f(x) = \left|1 - \left|1 - |x|\right|\right|$ is not differentiable at $x = 0, \pm 1, \pm 2$

2. **(d) :** If A has m members and B has n members,
$2^m - 2^n = 2016 \Rightarrow 2^n(2^{m-n} - 1) = 2^5(2^6 - 1)$
$\therefore \quad n = 5, m = 11.$
The number of members of $A \cup B$ is $11 + 5 - 3 = 13$.

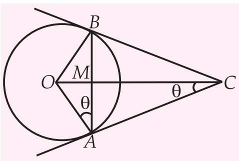
3. **(c) :** $S = \sum_{r=1}^{20} \frac{6 \times r!}{(r+3)!} = 6 \sum_{r=1}^{20} \frac{1}{(r+1)(r+2)(r+3)}$

$= \sum_{r=1}^{20} \left(\frac{3}{r+1} - \frac{6}{r+2} + \frac{3}{r+3} \right) = \frac{3}{2} - 2 + 1 - \frac{3}{22} + \frac{3}{23}$

$= \frac{125}{253} = \frac{m}{n}$
$\therefore \quad n - 2m = 3$

4. **(d) :** The girls can sit together in 1, 2; 2, 3 ; . . . . ; 11, 12 = 22 ways. If one boy sits between them, they sit in 1, 3; 2, 4; . . . . . .; 10, 12, = 20 ways
If two boys sit between them, they sit in 1, 4; 2, 5; .....; 9; 12 = 18 ways
The desired number is $12! - 60 \cdot 10! = \frac{6}{11} . 12!$

5. **(b) :**



$M = (2, 3), \quad OM = \sqrt{13}$
$OA = 5 \quad \Rightarrow \quad MA = \sqrt{25 - 13} = 2\sqrt{3}$
Area of $OACB = 2\, \Delta OAC = OA \cdot AC = 5 \cdot 5\cot\theta$

$= 25 \cdot \frac{2\sqrt{3}}{\sqrt{13}} = 50\sqrt{\frac{3}{13}}$

6. **(b) :** Using cosine rule,
$(x^2 - 1)^2 + (x^2 + x + 1)^2 - (2x + 1)^2 = \sqrt{3}(x^2 + x + 1)(x^2 - 1)$
$\Rightarrow \quad (2 - \sqrt{3})x^2 + (2 - \sqrt{3})x - (1 + \sqrt{3}) = 0,$
[Cancelling the factor $x^2 - 1$].

Solving, we get, $x = -(2 + \sqrt{3}), 1 + \sqrt{3}$
$x = -(2 + \sqrt{3})$ makes $c = 2x + 1 < 0$

7. **(d) :** $r_3 = s\tan\frac{C}{2} = s$
$\Delta^2 = r_1 r_2 r_3 r = r_1 r_2 \cdot \Delta$
$\Rightarrow \quad \Delta = \frac{1}{2}ab = 15$
$s - a = 3, s - b = 5$
$a - b = 2 \Rightarrow (a + b)^2 = (a - b)^2 + 4ab = 124$
$\therefore \quad a + b = 2\sqrt{31}$

8. **(c) :** $s = \sqrt{31} + 4, \; r = \frac{\Delta}{s} = \frac{15}{\sqrt{31} + 4} = \sqrt{31} - 4$

9. **(1) :** $t_n = \frac{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2n-1)}{7 \cdot 9 \cdot 11 \ldots (2n+5)}$

$= 15 \frac{1}{(2n+1)(2n+3)(2n+5)}$

$= \frac{15}{8} \left(\frac{1}{2n+1} - \frac{2}{2n+3} + \frac{1}{2n+5} \right)$

$\sum_{n=1}^{20} t_n = \frac{15}{8} \left(\frac{1}{3} - \frac{1}{5} - \frac{1}{43} + \frac{1}{45} \right) = \frac{32}{129} = \frac{m}{n}$

$\therefore \quad n - 4m = 1$

10. **(d) :** (P) $xy = \frac{1}{4}$, Now, let $x = \frac{t}{2}, y = \frac{1}{2t}$

Tangent $\frac{x}{2t} + \frac{t}{2}y = \frac{1}{2} \rightarrow OA = t, OB = \frac{1}{t}$
$OA \cdot OB = 1$
(Q) Tangent at $(\cos\theta, \sin\theta)$ is $x\cos\theta + y\sin\theta = 1$

$OA = \sec\theta, OB = \mathrm{cosec}\,\theta \rightarrow \frac{1}{OA^2} + \frac{1}{OB^2} = 1$

(R) Tangent at $(\cos^4\theta, \sin^4\theta)$ is
$x\sin^2\theta + y\cos^2\theta = \sin^2\theta\cos^2\theta$
$\therefore \quad OA = \cos^2\theta, OB = \sin^2\theta, OA + OB = 1$
(S) Tangent at $(\cos^3\theta, \sin^3\theta)$ is
$x\sin\theta + y\cos\theta = \sin\theta\cos\theta$
$OA = \cos\theta, OB = \sin\theta, AB = 1$

Solution Sender of Maths Musing

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